The sphere packing problem in dimensions 8 and 24

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Abstract

The sphere packing problem asks to find a packing of congruent spheres in $\mathbb{R}^n$ that has the biggest density among all possible sphere packings. We go through the 3 papers that led to the solution of this problem in dimensions 8 and 24. In 2003, Cohn and Elkies provided a new tool to prove upper bounds for the density of any sphere packing [1]. Building on their work, Viazovska proved in 2016 that the E_8 lattice packing is the densest sphere packing in 8 dimensions [2]. Shortly after and using the same methods, she and Cohn, Kumar, Miller and Radchenko proved that the Leech lattice packing is the densest sphere packing in 24 dimensions [3].

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1 Introduction

1.1 History and motivations

The problem of sphere packing, that is stacking unit spheres in the Euclidean space without overlap and try to get as little left space as possible, is an old mathematical question. In 1611, Johannes Kepler conjectured that the natural face-centered cubic packing and its variations provide the maximal density among all sphere packing in 3 dimensions. Carl Friedrich Gauss proved in 1831 that it was true among the lattice sphere packings, i.e. the packing where the centers of spheres form a lattice. Finally, in 1998, Thomas Hales provided a proof of the general case in a famous paper [4] of 250 pages coming with 3 gigabytes of computer programs and data. Due to the size of the proof, the referee's panel of 12 people that was in charge to evaluate the proof concluded only that they were "99%" sure that the proof was right. In the end, Hales and his collaborators provided a formal proof [5] in 2015 that can be totally checked by computers. Previously, the two-dimensional case, i.e. the packing of disks in the plane, was solved for lattice packings by Joseph-Louis Lagrange in 1773. The optimal one is given by the usual hexagonal lattice packing. The first complete proof in dimension 2 was done by László Fejes Tóth in 1940 [6].

For higher dimensions, not much was known until 2016. We know the optimal density of a lattice packing for dimensions less or equal to 8 and for dimension 24. But there is evidence that the densest sphere packing could be irregular. In particular, we know a non-lattice packing in $\mathbb{R}^{10}$ and in bigger dimensions that are denser than the best lattice packing known. However, for dimensions 8 and 24, a few evidence indicates that it could be simpler. There are good candidates for each dimension that have been known since a long time. They are the packing given by the E_8 lattice that we denote Λ_8, and the Leech lattice Λ_{24}. These lattices arise from different theories, especially from the theory of error-correcting code which is closely related to the sphere packing problem. The E_8 lattice is given by

$$\Lambda_8 := \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z}^8 + \tfrac{1}{2}\right)^8 \middle| \sum_{i=1}^{8} x_i = 0 \pmod 2 \right\}.$$

The Leech lattice is harder to describe explicitly. In the book *Sphere packings, lattices and groups* by Conway and Sloane [7] chapter 4, § 11 one can find various constructions and properties of the Leech lattice. We give matrices of generators in appendix A from which we can deduce all the required properties of the lattices.

These lattices arise from natural construction. For example, the 3-dimensional best sphere packing can be created in the following way: we embed copies of the usual 2-dimensional hexagonal packing in $\mathbb{R}^3$ and replace the circles by spheres. Then we put these copies over each other with the center of a new sphere exactly over the hollow between the spheres already in place. This construction can be continued in higher dimensions but it gives less and less good packings as the dimension grows. However, in dimension 8, the holes between spheres are getting just big enough to exactly put another one between them. This gives the E_8 lattice. A similar construction leads to the Leech lattice in 24 dimensions. Also, they are both unique in their respective dimension in some sense, and this uniqueness leads to the uniqueness of the optimal packing.

In 2003, Henry Cohn and Noam Elkies published an article [1], where they provided a new method to get an upper bound on the maximal density. This method is based on Poisson summation formula and trying to control some function and its Fourier transform at the same time, a very difficult task. For instance, the Heisenberg's uncertainty principle in physics is based on the inability to control together a function and its Fourier transform. Using linear programming to find a good candidate function, they improved the best bounds already known, that were found in a similar way. Especially, they got bound for dimensions 8 and 24 that were only 1.000001 and 1.0007071 times bigger than the density of the Λ_8 and Λ_{24} packings. Looking at these surprisingly close results, they began to suspect that some "magic" functions f_8 and f_{24} exist and give exactly the right density in dimensions 8 and 24. They began to seek for other properties of the magic functions. Collaborating with Kumar and later with Miller, they found lots of numerical evidence that a function satisfying the conditions was existing somewhere and that one just needs to find the right way to explicit it. As Cohn said [8]:

> "These calculations left no doubt that the magic functions existed: one could compute them to fifty decimal places, plot them, approximate their roots and power series coefficients, etc. They were perfectly concrete and accessible functions, amenable to exploration and experimentation, which indeed uncovered various intriguing patterns. All that was missing was an existence proof. However, proving existence was no easy matter. There was no sign of an explicit formula, or any other characterization that could lead to a proof. Instead, the magic functions seemed to come out of nowhere."

Considering the difficulty of the proof by Hales in dimension 3 and all the problems that could arise in larger dimensions, it was a real surprise that Maryna S. Viazovska created by announcing a short proof of the optimality of the E_8 lattice packing in an article [2] of only 24 pages. The very evening of her publication, Henry Cohn sent to her a mail proposing to apply her method to dimension 24 and just one week later, they were posting with 3 other collaborators an article proving the 24-dimension case [3]. The construction of the magic function by Viazovska relies on modular forms, special functions that have been known since a long time for their relations with lattices and for having deep and surprising relations with a lot of fields in mathematics. Moreover, Viazovska and its collaborators proved in 2019 that these two lattices are universally optimal [9], meaning that they minimize energy for a lot of potential functions. For example, putting electrons at each point of the lattices would give the lowest energy level among all configurations in these dimensions.

1.2 Overview of the plan

We start by introducing more rigorous definitions. Let $X \subseteq \mathbb{R}^n$ a discrete set such that for any two points $x, y \in X$, the Euclidean distance between them satisfies $|x - y| \geq 2$. We put unit balls $B_n(0,1)$ centered at each point of X. The condition above implies that the spheres have all disjoint interior. We call the set

$$P = \bigcup_{x \in X} B_n(x, 1)$$

a *sphere packing* of $\mathbb{R}^n$. If there is n linearly independent translations that left invariant the packing, P is said to be *periodic*. If X is a lattice, we say that P is a *lattice sphere packing*. The *density* of the packing P is defined as

$$\Delta_P := \limsup_{r \to \infty} \frac{\mathrm{Vol}(P \cap B_n(0,r))}{\mathrm{Vol}(B_n(0,r))},$$

where Vol is the Lebesgue measure on $\mathbb{R}^n$. Recall that

$$\mathrm{Vol}(B_n(0,r)) = \frac{\pi^{n/2} r^n}{\Gamma(\frac{n}{2} + 1)},$$

where Γ is the usual gamma function. Note also that the density does not change by rescaling or applying an isometry on the packing. So for any lattice, one can easily define the corresponding densest lattice packing by placing balls of radius half of the length of the shortest vector. For example, the E_8 lattice has a shortest vector of length $\sqrt{2}$ and the density of the E_8 lattice packing is $\frac{\pi^4}{384} \approx 0.25367$. Similarly, the Leech lattice has a minimal vector length of 2 and the density of the corresponding lattice packing is $\frac{\pi^{12}}{12!} \approx 0.00193$.

The sphere packing problem asks to find the maximal density Δ_P among all possible sphere packings P. For example, Hale proved that the maximal density in dimension 3 was $\frac{\pi}{\sqrt{18}} \approx 0.74048$ and Lagrange that the maximal density in dimension 2 is $\frac{\pi}{\sqrt{12}} \approx 0.90690$. The goal of this paper is to provide a mostly self-contained proof of the optimality of the E_8 lattice packing in dimension 8 and of the Leech lattice packing in dimension 24. We don't need too much information on these packings since we provide sharp bounds for the density. Hence, we only give computations of the density of these packings, and also use some other properties to prove uniqueness. We will begin by the proof the theorem of Cohn and Elkies (3.1 from [1]) which gives an upper bound on the density of any sphere packing.

Theorem 1.1 (Cohn, Elkies). *Let $f : \mathbb{R}^n \to \mathbb{R}$ a non-zero "admissible" function satisfying*

1. *For $|x| \geq 1$, $f(x) \leq 0$;*
2. *For any y, $\hat{f}(y) \geq 0$.*

Then the density of any sphere packing in dimension n is smaller than

$$\frac{f(0)}{2^n \hat{f}(0)} \operatorname{Vol}(B_n(0,1)).$$

Here, $\hat{f}$ means the Fourier transform of f and "admissible" is a convergence condition that we define later. The proof of the theorem consists of using the Poisson summation formula for lattices to make the density of a lattice packing appears and do estimations to bound the density from above. We will also discuss the uniqueness of the packings and give some useful properties of the functions that we seek. After that, we will review some basis about modular forms. In the end, we will prove the two main theorems of this paper:

Theorem 1.2 (Viazovska). *The density of a sphere packing in 8 dimensions is at most $\frac{\pi^4}{384}$ and the unique periodic packing achieving it is the E_8 lattice packing.*

Theorem 1.3 (Cohn, Kumar, Miller, Radchenko, Viazovska). *The density of a sphere packing in 24 dimensions is at most $\frac{\pi^{12}}{12!}$ and the unique periodic packing achieving it is the Leech lattice packing.*

We will proceed by constructing the respective magic functions for theorem 1.1 using modular forms. To this end, notice first that we can reduce ourselves to radial functions since the hypothesis and the conclusion of the theorem do not change under rotation. Hence, one can replace f by its average over rotations. Moreover, one can split f into eigenfunctions of the Fourier transform in the following way:

$$f_+ := \frac{f + \hat{f}}{2}, \; f_- := \frac{f - \hat{f}}{2}.$$

So f_+ and f_- are eigenfunctions of the Fourier transform of eigenvalues $+1$ respectively -1 and $f = f_+ + f_-$ and $\hat{f} = f_+ - f_-$. This allows us to split the construction of the magic functions into two distinct problems. Another important constraint on our construction is that the proof of theorem 1.1 tells us that the functions f and $\hat{f}$ must have roots at any non-zero point of the lattice if we want a sharp bound. So in particular, f_+ and f_- also share these roots. Hence, these two final chapters follow both the same process. First, we create two radial eigenfunctions of the Fourier transform with zeros at almost all points of the respective lattice. These functions consist of the product of the Laplace transform of a modular form and a squared sine factor that gives the right roots. Then we take a linear combination of these two forms to get the desired functions for theorem 1.1 and we prove that they satisfy the right properties to conclude.

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2 The linear programming bound

The bound of theorem 1.1 is usually called the linear programming bound, even if it does not rely on linear programming at all. This is because one can use linear programming tools to look for functions that gives the best bounds, or at least approximations of the best bounds. The proof of this theorem relies on a version of the Poisson summation formula for lattices. This chapter begins by some review of Fourier analysis and lattices in $\mathbb{R}^n$ and the proof of the Poisson summation formula. Then we prove the main theorem 1.1 of Cohn and Elkies and we conclude by discussing some properties required by the magic functions to give a sharp bound and uniqueness.

2.1 Fourier analysis and lattices in $\mathbb{R}^n$

We work with the following normalization of Fourier transform and we reduce ourselves to nice-behaved functions for convergence purposes.

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ an $L^1(\mathbb{R}^n)$ function. The *Fourier transform* of f is

$$\mathcal{F}(f)(y) = \hat{f}(y) := \int_{\mathbb{R}^n} f(x)e^{-2i\pi\langle x,y\rangle}dx$$

for any $y \in \mathbb{R}^n$.

Definition 2.2. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is *admissible* if there exist constants $\delta > 0$ and C such that both $|f(x)|$ and $|\hat{f}(x)|$ are bounded by $C(1 + |x|)^{-n-\delta}$. f is a *Schwartz function* if $f \in C^\infty(\mathbb{R}^n)$ and all derivatives of f are $O\left((1 + |x|)^{-k}\right)$ for all k positive integer.

Remark. Note that a Schwartz function is always admissible because Fourier transform maps Schwartz functions to Schwartz functions. Moreover, if f is radial, then it is a Schwartz function if and only if the function $f(r) := f(|x|)$ is Schwartz, where $x \in \mathbb{R}^n$ is of norm r. Most of the functions we will consider are radial Schwartz functions. Also, Fourier inversion theorem is valid for admissible functions since their Fourier transform is L^1.

Definition 2.3. Let $\Lambda \subseteq \mathbb{R}^n$ a lattice. We denote by $|\Lambda|$ the determinant of a matrix of generators of Λ which is also the volume of a fundamental parallelogram of Λ. The *dual lattice* Λ^* of Λ is

$$\Lambda^* := \{t \in \mathbb{R}^n | \forall x \in \Lambda \ \langle t, x\rangle \in \mathbb{Z}\}.$$

Note that a basis of Λ^* is given by the dual basis of any basis of Λ. Hence, $|\Lambda^*| = |\Lambda|^{-1}$. If $|\Lambda| = |\Lambda^*| = 1$, the lattice is called *unimodular*. A lattice is *integral* if the inner product of two vectors of the lattice is always an integer. It is *even* if the squared norm of a vector is always even. The *Gram matrix* of a basis of Λ is the matrix of the basis times its transpose. It gives the inner product between elements of the basis. In particular, Λ is integral, respectively even if and only if the Gram matrix has integer respectively even entries.

Lemma 2.4. *Let Λ an integral unimodular lattice. Then $\Lambda^* = \Lambda$.*

Proof. Since Λ is integral, we have $\Lambda \subseteq \Lambda^*$. And since Λ is also unimodular, we must have equality, because then they both have the same fundamental parallelogram. □

Remark. From the matrix of generators of Λ_8 and Λ_{24} given in appendix A, one can easily compute its *Gram matrix*, which consists of all the inner product between vectors of the basis and is equal to the product of the matrix and its transpose. These two matrices implies that the two lattices are integral, even and unimodular. Hence, we also have that $\Lambda_8^* = \Lambda_8$ and $\Lambda_{24}^* = \Lambda_{24}$.

Definition 2.5. Let $f : \mathbb{R}^n \to \mathbb{R}$ a Λ-periodic function, i.e. for all $x \in \mathbb{R}^n$ and $v \in \Lambda$, $f(x+v) = f(x)$. The *Fourier coefficients* of f corresponding to Λ are given by

$$f_t := \int_{\mathbb{R}^n/\Lambda} f(x) e^{-2i\pi\langle t,x\rangle} dx$$

for $t \in \Lambda^*$, where the integral is given by the projection of the Lebesgue measure on $\mathbb{R}^n$ to $\mathbb{R}^n/\Lambda$.

Theorem 2.6. *Let $f : \mathbb{R}^n \to \mathbb{R}$ a Λ-periodic and $L^1(\mathbb{R}^n/\Lambda)$ function such that the sum of the Fourier coefficients f_t, $t \in \Lambda^*$ converges absolutely. Then*

$$f(x) = \frac{1}{|\Lambda|} \sum_{t\in\Lambda^*} f_t e^{2i\pi\langle t,x\rangle}$$

almost everywhere (with respect to Lebesgue measure). Moreover, the function on the right-hand side of the equality is continuous.

Proof. By hypothesis, $\frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} f_t e^{2i\pi\langle t,x\rangle}$ is a uniformly convergent series of continuous functions, hence it is also continuous. We consider the function $g(x) := f(x) - \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} f_t e^{-2i\pi\langle t,x\rangle}$ and compute its Fourier coefficients. Let $u \in \Lambda^*$ and g_u the corresponding Fourier of g.

$$g_u = \int_{\mathbb{R}^n/\Lambda} \left(f(x) - \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} f_t e^{2i\pi\langle t,x\rangle}\right) e^{-2i\pi\langle u,x\rangle} dx = f_u - \frac{1}{|\Lambda|}\sum_{t\in\Lambda^*} f_t \int_{\mathbb{R}^n/\Lambda} e^{2i\pi\langle t-u,x\rangle} dx.$$

Since the sum of the Fourier coefficients converges absolutely, it is valid to exchange the sum and the integral. The value of the integral is

$$\int_{\mathbb{R}^n/\Lambda} e^{2i\pi\langle t-u,x\rangle} dx = \int_{[0,1]^n} |\Lambda| e^{2i\pi\langle t-u,y\rangle} dy = |\Lambda| \prod_{j=1}^{n} \int_0^1 e^{2i\pi(t_j-u_j)y_j} dy_j = \begin{cases} |\Lambda| & \text{if } t = u, \\ 0 & \text{else.} \end{cases}$$

So $g_u = 0$, hence all the Fourier coefficients of g are zeros. On the other hand, the trigonometric polynomials are dense in $C(\mathbb{R}^n/\Lambda)$ by the Stone-Weierstrass theorem (see for example [10], theorem 2.40), therefore also in $L^1(\mathbb{R}^n/\Lambda)$. So a function in $L^1(\mathbb{R}^n/\Lambda)$ with all Fourier coefficients equal to zero is null almost everywhere. Hence, the theorem holds. □

Theorem 2.7 (Poisson summation formula)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ a lattice, $y \in \mathbb{R}^n$ a vector and $f : \mathbb{R}^n \to \mathbb{R}$ an admissible function. Then*

$$\sum_{x\in\Lambda} f(x+y) = \frac{1}{|\Lambda|} \sum_{t\in\Lambda^*} \hat{f}(t) e^{-2i\pi\langle y,t\rangle}.$$

Proof. We base our proof on the book *Introduction to Fourier Analysis on Euclidean Spaces* by Stein and Weiss [11]. The two sums converge absolutely by hypothesis. Moreover, the first sum is a periodic function with respect to Λ. We prove that its Fourier series is the second sum. Let $t \in \Lambda^*$, g_t the corresponding Fourier coefficient of the first sum and P a fundamental parallelogram of Λ:

$$\begin{aligned} g_t &= \int_P \left(\sum_{x\in\Lambda} f(x+y) \right) e^{-2i\pi\langle y,t\rangle} dy = \sum_{x\in\Lambda} \int_P f(x+y) e^{-2i\pi\langle y,t\rangle} dy \\ &= \sum_{x\in\Lambda} \int_{P-x} f(y) e^{-2i\pi\langle y-x,t\rangle} dy = \int_{\mathbb{R}^n} f(y) e^{-2i\pi\langle y,t\rangle} dy = \hat{f}(t). \end{aligned}$$

Interchanging integral and sum is valid by absolute convergence and the third equality comes from the definition of fundamental domain. Theorem 2.6 implies then that

$$\sum_{x\in\Lambda} f(x+y) = \frac{1}{|\Lambda|} \sum_{t\in\Lambda^*} \hat{f}(t) e^{-2i\pi\langle y,t\rangle}$$

almost everywhere. But since both functions are continuous, they are equal everywhere. □

2.2 Proof of the theorem of Cohn and Elkies

Proof of 1.1. We start by reducing the theorem to the case of *periodic packing*, i.e. packing where the center of the spheres are given by finitely many translations of a lattice Λ. The density of such packings can come arbitrary close to the density of any packing. To see this, let P an arbitrary set of points forming a packing. We approximate this packing by taking a finite portion of it and tiling the whole space with it. More precisely, consider the set of all spheres of P included in $[-a, a]^n$ for $a > 0$. This gives finitely many spheres of centers $v_1, \ldots, v_{N_a}$. Consider the packing

$$\tilde{P}_a = \bigcup_{i=1}^{N_a} (\Lambda_a + v_i),$$

where $\Lambda_a = 2a\mathbb{Z}^n$. One can see that the density of $\tilde{P}_a$ converges to the density of P as a goes to infinity. This is because one can also compute the density using intersection with a growing box instead of a ball, and because the number of balls intersecting the boundary of the box is negligible with respect to the number of spheres inside the box as r goes to infinity. For more details about the equivalent definitions of density, see appendix A in [1].

Now, we consider a periodic packing given by a lattice Λ and N vectors $v_1, \dots, v_N$. In this case, one easily see that we can reduce the computation of the density to any fundamental domain of the lattice. Therefore, the density of the packing is

$$\Delta = \frac{N \operatorname{Vol}(B_n(0, r/2))}{|\Lambda|},$$

where r is the minimal distance between two centers of the packing, because for any translate of the fundamental domain, there is exactly N balls in the packing.

Without loss of generality, we rescale the packing such that the minimal distance between two centers is 1. Let f an admissible function satisfying the conditions of the theorem. The Poisson summation formula tells us that

$$\sum_{j,k=1}^{N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{j,k=1}^{N} \sum_{t \in \Lambda^*} e^{-2i\pi \langle v_j - v_k, t \rangle} \hat{f}(t) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t) \left| \sum_{j=1}^{N} e^{-2i\pi \langle v_j, t \rangle} \right|^2.$$

Note that since the minimal distance between two spheres is 1, $|x + v_j - v_k| < 1$ if and only if $x = 0$ and $j = k$. Hence, using that $f(x) \leq 0$ for $|x| > 1$, we can bound the left side from above by $Nf(0)$. Also, using that $\hat{f}(x) \geq 0$ for all x, we can bound the right side from below by $\frac{N^2 \hat{f}(0)}{|\Lambda|}$. Therefore, we have

$$Nf(0) \geq \sum_{j,k=1}^{N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t) \left| \sum_{j=1}^{N} e^{-2i\pi \langle v_j, t \rangle} \right|^2 \geq \frac{N^2 \hat{f}(0)}{|\Lambda|}.$$

Hence, we conclude that

$$\frac{f(0)}{\hat{f}(0)} \geq \frac{N}{|\Lambda|}.$$

Multiplying by $\operatorname{Vol}(B_n(0, 1/2))$ on each side, we conclude the proof of the theorem. □

We also prove a modified version of this theorem that is more useful for applications:

Theorem 2.8 (Cohn, Elkies)**.** *Let $r > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ a non-zero admissible function satisfying*

1. $f(0) = \hat{f}(0) = 1$;
2. *For* $|x| \geq r$, $f(x) \leq 0$;
3. *For any* y, $\hat{f}(y) \geq 0$.

Then the density of any sphere packing in dimension n is smaller than

$$\left(\frac{r}{2}\right)^n \operatorname{Vol}(B_n(0, 1)). \tag{2.1}$$

Proof. Notice that $g : x \mapsto f(\frac{x}{r})$ satisfies the conditions of the theorem. Using usual properties of Fourier transform, we know that

$$\hat{g}(0) = \int_{\mathbb{R}^n} f(\tfrac{x}{r}) dx = \frac{1}{r} \int_{\mathbb{R}^n} f(x) dx = \frac{1}{r} \hat{f}(0).$$

We conclude by applying theorem 1.1 to g. □

Remark. Looking at the proof of theorem 1.1, we can immediately gather information about the magic functions we are looking for. For the inequalities to be sharp, one needs to make no error by deleting the terms at all the non-zero points of the lattice and its dual. Hence, f and $\hat{f}$ must have zeros at all non-zero points of the lattice, respectively the dual lattice. Moreover, since f and $\hat{f}$ do not change sign outside of the unit ball, all these zeros must be double zeros (or even of bigger degree), except the ones of f at $|x| = r$, where they must be of degree 1 (or of odd degree). In this case, the Poisson summation formula says that $f(0) = \hat{f}(0)$ as in theorem 2.8.

Remark. In our case, we know that the E_8 and Leech lattices are their own dual. Moreover, Λ_8 has vector length $\sqrt{2k}$ for $k = 1, 2, 3, \dots$ and Λ_{24} for $k = 2, 3, \dots$. This is easily seen for Λ_8 since the second vector of the matrix has norm $\sqrt{2}$. For Λ_{24}, one can prove this using modular forms (see [7], chapter 12). As we said in the introduction, we can suppose that the magic functions are radial. We write $f(r)$ for the value of f at any x of norm r.

In view of this information, Cohn and Miller made numerical experiments by combining functions of the form $p(r)e^{-\pi r}$, where p is a polynomial, to approximate the magic functions. From these, they gathered other information on it, especially that their second Taylor coefficient was probably rational. More precisely:

$$f_8(r) = 1 - \frac{27}{10}r^2 + O(r^4) \qquad \hat{f}_8(r) = 1 - \frac{3}{2}r^2 + O(r^4)$$
$$f_{24}(r) = 1 - \frac{14347}{5460}r^2 + O(r^4) \qquad \hat{f}_{24}(r) = 1 - \frac{205}{156}r^2 + O(r^4)$$

2.3 Uniqueness

We now give the proof that the best sphere packings are unique in dimensions 8 and 24 given the existence of the magic functions. Of course, these optimal packings can not be unique since removing one sphere from them would not change the density. But if we reduce ourselves to periodic packing, then it is the case. This relies on the uniqueness of the E_8 and Leech lattices up to certain conditions and on some properties of the magic functions that we are looking for.

Proposition 2.9. *Assume there exist functions f_8 and f_{24} satisfying the hypothesis of theorem 2.8 in dimension 8 and 24 and such that the bound given by equation (2.1) on the density equal the density of the E_8 packing, respectively the Leech packing. If in addition these functions vanish only at the vector lengths of the respective lattice, then there is a unique optimal periodic sphere packing corresponding to this lattice.*

To prove proposition 2.9, we begin with a small lemma on subgroups of $\mathbb{R}^n$.

Lemma 2.10. *Let S a subset of $\mathbb{R}^n$ such that S contains 0 and n linearly independent vectors. Assume that the squared distance between any two points of S is always an even integer. Then the additive subgroup of $\mathbb{R}^n$ generated by S is an even integral lattice.*

Proof. Let $G \subseteq \mathbb{R}^n$ be the subgroup generated by S. Then for any $x, y \in S$, we have

$$2\langle x, y\rangle = |x - 0|^2 + |y - 0|^2 - |x - y|^2,$$

and the numbers on the right-hand side are even and squares, so the inner product of two vectors in S is an integer. This extends easily to G by linearity. This implies that the squared distance between two vectors of G is also an integer, hence G is an integral lattice (because it contains a basis of $\mathbb{R}^n$). Finally, the squared norm of a vector is an even integer by hypothesis, so this lattice is even. $\square$

Proof of proposition 2.9. Now, let us consider a periodic packing P of $\mathbb{R}^n$, for $n = 8$ or 24 which has the same density as Λ_8 or Λ_{24}. This packing is given by finitely many translates of a lattice Λ by vectors $v_1, \ldots, v_N$. Note that rescaling and translating Λ do not change the density of the sphere packing. Thus, without loss of generality, we can suppose that $v_1 = 0$ and $|\Lambda| = N$. Applying the Poisson summation formula for $f = f_8$ or f_{24} as in the proof of the linear programming bound 1.1, we conclude that

$$Nf(0) \geq \sum_{j,k=1}^{N} \sum_{x \in \Lambda} f(x + v_j - v_k) = \frac{1}{|\Lambda|} \sum_{t \in \Lambda^*} \hat{f}(t) \left| \sum_{j=1}^{N} e^{-2i\pi \langle v_j, t \rangle} \right|^2 \geq \frac{N^2 \hat{f}(0)}{|\Lambda|}.$$

By the condition $f(0) = \hat{f}(0)$ from theorem 2.8 and $|\Lambda| = N$, the two bounds are equal. So there must be equality everywhere. In particular, each non-zero point of the packing $x + v_j - v_k$ must occur at a zero of f, hence it sits inside the E_8 or the Leech lattice by the additional conditions on f. So if we look at the set of points of our periodic packing, it satisfies the hypothesis of lemma 2.10 and generates a subgroup of $\mathbb{R}^n$ which is an even integral lattice Λ' included inside Λ_8 or Λ_{24}.

Since the lattice is integral, its Gram matrix is an integral matrix and has an integral determinant. Hence, $|\Lambda'| \geq 1$ since it is the square root of this integer. Thus, Λ' has at most one point per unit of volume in $\mathbb{R}^n$. But since $|\Lambda| = N$, our periodic packing P, which is given by

$$P = \bigcup_{i=1}^{N} (\Lambda + v_i),$$

has already one sphere per unit of volume. So in fact, Λ' cannot be bigger than P since both are periodic. Hence, the sphere packing is a lattice packing given by Λ'.

Finally, to have the density of the E_8, respectively Leech packing, the minimal vector length of Λ' (which gives the size of the packing) must be the square root of 2, respectively 4. Indeed, the density of the packing Λ' is

$$\mathrm{Vol}(B_n(0, r/2)) = \frac{\pi^{n/2} r^n}{2^n \Gamma(n/2 + 1)},$$

where r is the minimal vector length, and the density of the respective lattices are $\frac{\pi^4}{384}$ and $\frac{\pi^{12}}{12!}$ as seen before. As it is stated below in theorem 2.11, Λ_8 and Λ_{24} are the unique unimodular even lattices in dimensions 8 and 24 with such minimal vector lengths. Since Λ' satisfies the same conditions, we conclude that it is equal to Λ_8 or Λ_{24}. So the E_8 lattice packing and the Leech lattice packing are unique in their respective dimension. $\square$

Theorem 2.11. *The E_8 lattice is the unique lattice Λ in $\mathbb{R}^8$ (up to isometries) such that:*

1. *Λ is unimodular;*

2. *Λ is even.*

The Leech lattice is the unique lattice Λ in $\mathbb{R}^{24}$ (up to isometries) such that:

1. *Λ is unimodular;*

2. Λ *is even;*

3. *The minimal non-zero vector of* Λ *has norm 2.*

Proof. See [7], chapters 16 and 18. □

3 Modular forms

Modular forms are analytic functions of the upper half complex plan which transform in a certain way under the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$ and which do not grow too fast at $i\infty$. It turns out that these functions have a lot of good behaviors and applications, in particular in number theory. Moreover, it is possible to generalize the definition of modular form in various ways. For example, by using other discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$ or by relaxing the growth condition.

3.1 Introduction to modular forms

We begin by an overview of the relation between the upper half plan and the group $\mathrm{SL}(2,\mathbb{R})$, and of the definition of modular forms. For a complete introduction to these concepts, see [12] or [13].

Let $\mathbb{H} := \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ be the *upper half plan* of the complex plan. We consider on it the action of $\mathrm{SL}(2,\mathbb{R})$ by *linear fractional transformations*, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d},$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}$. This action is well defined on $\mathbb{H}$, since

$$Im\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \operatorname{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2}. \tag{3.1}$$

We consider the restriction of this action to the subgroup $\mathrm{SL}(2,\mathbb{Z})$ of matrices with integer coefficients, called the *full modular group.* This group is generated by the two matrices:

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For a positive integer N, the *principal congruence subgroup of level* N is defined as

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \middle| a = d = 1 \pmod N,\ b = c = 0 \pmod N \right\}.$$

A subgroup $\Gamma \subseteq \Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$ is a *congruence subgroup* if there is some positive integer N such that $\Gamma(N) \subseteq \Gamma$. The *factor of automorphy* is defined by

$$j(z,\gamma) := cz+d,$$

for all $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. It satisfies the following *chain rule*:

$$j(z,\gamma_1\gamma_2) = j(z,\gamma_2)j(\gamma_2 z,\gamma_1).$$

Let $f : \mathbb{H} \to \mathbb{C}$, $\gamma \in \Gamma(1)$ and $k \in \mathbb{Z}$. We denote by $f[\gamma]_k$ the function defined by

$$f[\gamma]_k(z) := j(z,\gamma)^{-k} f(\gamma z).$$

The chain rule implies that $f[\gamma_1\gamma_2]_k = (f[\gamma_1]_k)[\gamma_2]_k$. Hence, this operator is an action of $\Gamma(1)$ on the complex-valued functions on $\mathbb{H}$, called the *weight-k operator*. Moreover, for a positive integer N, the matrix $T^N := \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ belongs to $\Gamma(N)$ and acts as

$$f[T^N]_k(z) = f(z+N).$$

Hence, if f is invariant under $\Gamma(N)$, it must be N-periodic. If it is also holomorphic, it admits a Fourier series of the form

$$f(z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} f_n(y) q^n,$$

where $z = x + iy$ and $q = e^{2i\pi z}$. This is standard notations which we will not recall each time. One can show that f_n is independent of y. We can now define a modular form.

Definition 3.1. Let k an integer and Γ a congruence subgroup containing $\Gamma(N)$ for some N. A *(holomorphic) modular form* of weight k and group Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. $f[\gamma]_k = f$ for all $\gamma \in \Gamma$;
2. For all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, the Fourier expansion of $f[\gamma]_k$ is of the form

$$f[\gamma]_k(z) = \sum_{n=0}^{\infty} f_{\gamma,\frac{n}{N}} e^{2i\pi \frac{n}{N} z}.$$

If in addition $f_{\gamma,0} = 0$ for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, f is called a *cusp form*. The space of modular forms of weight k and congruence subgroup Γ is denoted $\mathcal{M}_k(\Gamma)$ and the subspace of cusp forms $\mathcal{S}_k(\Gamma)$. Since the multiplication of two modular forms gives another modular form of weight the sum of the two weights, we can define the graded ring $\mathcal{M}(\Gamma) := \oplus_{k>0}\mathcal{M}_k(\Gamma)$ of modular forms associated to Γ. The subring $\mathcal{S}(\Gamma) := \oplus_{k>0}\mathcal{S}_k(\Gamma)$ of cusp forms is an ideal of $\mathcal{M}(\Gamma)$.

Remark.
1. For a modular form of weight k on the full modular group $\Gamma(1)$, one can reduce himself to test the first condition only on S and T since these two matrices generate $\Gamma(1)$. Hence, it is enough to compute that

$$f(z) = f(Tz) = f(z+1), \qquad f(z) = z^{-k} f(Sz) = z^{-k} f(-1/z).$$

2. For the second condition, we know that $f[\gamma]_k$ admits a Fourier series from the first one. Hence, we just have to check that the limit of $f[\gamma]_k(z)$ when $\mathrm{Im}(z) \to \infty$ exists. In particular, f is a cusp form if and only if $f((z) \to 0$ when $\mathrm{Im}(z) \to \infty$.
3. The second condition is called *holomorphy at the cusps*. We explain this terminology. We consider the completion of $\mathbb{H}$ given by $\bar{\mathbb{H}} := \{z \in \mathbb{C} | \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. The *cusps* are the points $\mathbb{Q} \cup \{\infty\}$. We extend the action of $\mathrm{SL}(2,\mathbb{R})$ to this completion by continuity. It gives the same action on $\mathbb{R}$ when the denominator does not vanish and behaves in a natural

way at infinity. Intuitively, the holomorphy at infinity is given by f is holomorphic when $\mathrm{Im}(z) \to \infty$. To be more rigorous, note that the map $z \mapsto e^{2i\pi z/N}$ is an $N\mathbb{Z}$-periodic map sending $\mathbb{H}$ to the open punctured unit disk centered at 0. With respect to our completion, it sends $\mathbb{R}$ to the unit circle S^1 and ∞ to 0. This function has an inverse given by $q \mapsto \frac{N\log(q)}{2i\pi}$. In this context, holomorphy at infinity only means that $f\left(\frac{N\log(q)}{2i\pi}\right)$ is holomorphic at 0. For the full modular group $\mathrm{SL}(2,\mathbb{Z})$, this condition suffices since the linear transformation $z \mapsto \frac{az+b}{cz+d}$ send ∞ to $\frac{a}{c}$ for any $\frac{a}{c} \in \mathbb{Q}$. And the existence of $b, d \in \mathbb{Z}$ is guaranteed for a and c coprime by Euclid's algorithm. But for congruence subgroups, there can be more than one Γ-equivalence class of cusps, so one need to look at holomorphy at each of them.

4. The spaces $\mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k(\Gamma)$ are all of finite dimension. This is one of the reasons for the good behavior of modular forms, because it makes coincidences frequent. Moreover, for negative k, these spaces are equal to $\{0\}$. Also, for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ or $\Gamma(2)$, the space is also empty for odd k. This is because of the matrix $-Id \in \mathrm{SL}(2,\mathbb{Z})$. Hence, for $f \in \mathcal{M}_k(\Gamma)$, we have $f = f[-Id]_k = (-1)^k f$, so $f = 0$.

A natural way to extend the definition above is to allow poles of finite order at the cusps, by making the Fourier series begin at a negative integer. This leads to the following definition:

Definition 3.2. Let k an integer and Γ a congruence subgroup containing $\Gamma(N)$ for some N. A *weakly holomorphic modular form* of weight k and group Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that:

1. $f[\gamma]_k = f$ for all $\gamma \in \Gamma$;

2. For all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, the Fourier expansion of $f[\gamma]_k$ is of the form

$$f[\gamma]_k(z) = \sum_{n=n_0}^{\infty} f_{\gamma,\frac{n}{N}} e^{2i\pi\frac{n}{N}z},$$

for some fixed $n_0 \in \mathbb{Z}$.

The space of weakly holomorphic modular forms of weight k and group Γ is denoted $\mathcal{M}_k^!(\Gamma)$.

Remark. Unlike the space of modular forms, the space of weakly holomorphic modular forms is infinite dimensional. But in return, there exist non-zero weakly holomorphic modular forms of negative weight.

Another natural way to extend the definition of modular forms is to extend its series, not with an expansion giving poles at cusps, but with a finite expansion in $\mathrm{Im}(z)^{-1}$.

Definition 3.3. Let k an integer and Γ a congruence subgroup. An *almost holomorphic modular form* of weight k and group Γ is a polynomial $f(z) = \sum_{j=1}^{r} f_j(z)y^{-j}$ in $y = \mathrm{Im}(z)$ such that:

1. $f[\gamma]_k = f$ for all $\gamma \in \Gamma$.

2. Each f_j is a holomorphic function which is holomorphic at all cusps.

r is the *depth* of f (if it is minimal, i.e. $f_r \neq 0$). A *quasimodular form* is the constant term f_0 of an almost holomorphic modular form f. The space of quasimodular form of weight k, depth r and group Γ is denoted $\widetilde{\mathcal{M}}_k^{(r)}(\Gamma)$. We also have the graded $\widetilde{\mathcal{M}}(\Gamma) := \bigoplus_k \bigcup_{r\geq 0} \widetilde{\mathcal{M}}_k^{(\leq r)}(\Gamma)$ of all quasimodular forms on the group Γ.

Proposition 3.4. *Let ϕ a quasimodular form of weight 2 on Γ which is not modular. Then every quasimodular form on Γ is a polynomial in ϕ with modular forms as coefficients. More precisely, for all $k, p \geq 0$,*

$$\widetilde{\mathcal{M}}_k^{(\leq r)}(\Gamma) = \bigoplus_{j=0}^{r} \mathcal{M}_{k-2j}(\Gamma)\phi^j.$$

Proof. Let $f_0 \in \widetilde{\mathcal{M}}_k^{(\leq r)}(\Gamma)$ an almost holomorphic modular form and $f(z) = \sum_{j=0}^{r} f_j(z)y^{-j}$ the corresponding almost holomorphic modular form. Equation 3.1 tells us that for all $\gamma \in \Gamma$ and $j = 0, \dots r$,

$$f_j[\gamma]_k(z) = (f_j(z)y^{-j})[\gamma]_k = j(z,\gamma)^{-k} f_r(\gamma z)\frac{|j(z,\gamma)|^{2r}}{y^r} = j(z,\gamma)^{-k} f_r(\gamma z)\left(\frac{(cz+d)^2}{y} - 2iyc(cz+d)\right)^r$$

Expanding the factor in r, we get the term $j(z,\gamma)^{2j-k}y^{-j}f_j(\gamma z)$ and other terms of lower power in y^{-1}. Note that the y^{-r} term in $f[\gamma]_k$ is $j(z,\gamma)^{2r-k}f_r(\gamma)$. Hence, since $f[\gamma]_k = f$, f_r must be a modular form of weight $k - 2r$ for Γ. In particular, $r \leq k/2$ because there is no modular form of negative weight.

Let ϕ^* the almost holomorphic modular form corresponding to ϕ. By hypothesis, $1 \leq r \leq 2/2$ for ϕ^*. Hence, ϕ^* is the sum of ϕ and a non-zero constant times $1/y$. So we can subtract from f a multiple of $(\phi^*)^r f_r$ to cancel the f_r term and get an almost holomorphic modular form of smaller depth. Therefore, the statement follows by induction on the depth r. □

Remark. We can also mix the definitions above to get *weakly holomorphic quasimodular forms*. They are the constant term of a polynomial in $\mathrm{Im}(z)^{-1}$ with coefficients that are functions with poles of finite order at the cusps (and the polynomial behaves like a modular form under the action of a congruence subgroup Γ).

It will be important, for convergence purposes, to have bounds on the Fourier coefficients.

Theorem 3.5. *Let f a weakly holomorphic quasimodular form and $f(z) = \sum_{n=n_0}^{\infty} f_n q^n$ its Fourier series. Then, for all n, we have*

$$f_n = O\left(e^{C\sqrt{|n_0 n|}}\right),$$

with the constant depending on the number of cusps, k and the depth of f.

Remark. References for this result on weakly holomorphic modular forms are given in [14]. It also gives a more precise formula. The more general result for quasimodular forms is a formula of the same shape.

3.2 Eisenstein series

Definition 3.6. Let $k \geq 4$ an even integer. The *weight k Eisenstein series* is defined as

$$E_k(z) := \frac{1}{2\zeta(z)} \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(mz+n)^k}.$$

Here, ζ denote the Riemann zeta function and the factor $\frac{1}{2\zeta(z)}$ is a normalization factor. It is a modular form of weight k for $\Gamma(1) = \mathrm{SL}(2,\mathbb{Z})$. Since $k > 2$, the sum converges absolutely so we do not need to specify an order of summation on m and n.

To prove that $E_k[\gamma]_k = E_k$ for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$, we can reduce ourselves to show it for the matrices S and T. For S, one computes

$$E_k(Sz) = \frac{1}{2\zeta(z)} \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(-m\frac{1}{z}+n)^k} = z^k \frac{1}{2\zeta(z)} \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(-m+nz)^k} = z^k E_k(z).$$

For T, the result is obvious since, for a fixed m, it just translate the summation on n by m. The holomorphy at infinity is also clear since the absolute convergence implies that all terms of the sum vanish except for $n = 0$ when $\mathrm{Im}(z) \to \infty$.

Proposition 3.7. *The Fourier series of the Eisenstein series is*

$$E_k(z) = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

with $\sigma_k(n) := \sum_{d|n} d^k$.

Proof. We use the following representation of the cotangent function:

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right).$$

A nice elementary proof of this formula is given in [15]. Denoting $q = e^{2i\pi z}$, we also have

$$\pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)} = -i\pi \frac{1+q}{1-q} = -i\pi - 2i\pi \sum_{n=1}^{\infty} q^n.$$

Differentiating $k-1$ times these two expressions, we get

$$(-1)^{k-1}(k-1)! \left(\frac{1}{z^k} + \sum_{n=1}^{\infty} \left(\frac{1}{(z-n)^k} + \frac{1}{(z+n)^k} \right) \right) = -(2i\pi)^k \sum_{n=1}^{\infty} n^{k-1} q^n$$

We rewrite this as

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2i\pi)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

Hence, for even k,

$$\sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(mz+n)^k} = 2\zeta(k) + 2\sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} = 2\zeta(k) + 2\sum_{m=1}^{\infty} \frac{(2i\pi)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^{mn}$$

Rearranging the terms, we get that

$$E_k(z) = 1 + \frac{(2i\pi)^k}{(k-1)!\zeta(k)} \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m.$$

We conclude by applying the functional equation of the zeta function:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Note that $\sin\left(\frac{\pi k}{2}\right)\Gamma(1-k) = \frac{(-1)^{k/2}\pi}{2}\frac{(-1)^k}{(k-1)!} = \frac{i^k \pi}{2(k-1)!}$ by cancellation of the zero of the sine and the pole of the gamma function. $\square$

Definition 3.8. This series also make sense for $k = 2$, so we define

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) e^{2i\pi nz}.$$

This function is not modular. However, it is $\mathbb{Z}$-periodic and satisfies

$$z^{-2}E_2(-1/z) = E_2(z) - \frac{6i}{\pi z}. \tag{3.2}$$

A proof of this formula is given in [12], chapter 2.3 of Zagier's part. This is an example of a quasimodular form. One of the most important properties of the Eisenstein series is that they give a basis of the modular forms for the full modular group $\Gamma(1)$.

Theorem 3.9. *The graded ring $\mathcal{M}(\mathrm{SL}(2,\mathbb{Z}))$ is isomorphic to $\mathbb{C}[E_4, E_6]$ as a $\mathbb{C}$-algebra.*

Proof. See [12], proposition 4, chapter 2 of Zagier's part. □

Corollary 3.10. *E_2 is a quasimodular form of weight 2 and depth 1, and the graded ring $\widetilde{\mathcal{M}}(\mathrm{SL}(2,\mathbb{Z}))$ is isomorphic to $\mathbb{C}[E_2, E_4, E_6]$.*

Proof. E_2 is the constant term of the almost holomorphic modular form $f(z) := E_2(z) - \frac{3}{\pi y}$. $f(z)$ is clearly 1-periodic and using the equation 3.1, we get that

$$z^{-2}f(-1/z) = E_2(z) - \frac{6i}{\pi z} - \frac{3|z|^2}{\pi y z^2} = E_2(z) - \frac{6i}{\pi z} - \frac{3\bar{z}}{\pi y} = E_2(z) - \frac{6i}{\pi z} + \frac{6iy}{\pi y z} - \frac{3z}{\pi y z} = f(z).$$

The second statement follows directly from the theorem and proposition 3.4. □

Remark. This corollary allows us to understand how a quasimodular form on $\mathrm{SL}(2,\mathbb{Z})$ behaves. Since E_2 is 1-periodic, as well as E_4 and E_6. So is any quasimodular form on $\Gamma(1)$. Moreover, if $f \in \widetilde{\mathcal{M}}_k^{(r)}(\Gamma(1))$ is a quasimodular form of weight k and depth r, then $f(-1/z)z^{-k}$ is equal to the sum of $f(z)$ and some other factors in $\frac{1}{z}, \dots, \frac{1}{z^r}$ coming from the equation 3.2 for $E_2(-1/z)z^{-2}$.

Definition 3.11. The discriminant function is defined by

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

with $q = e^{2i\pi z}$. For $z \in \mathbb{H}$, we have $|e^{2i\pi z}| < 1$, so by usual properties of Euler products, the product converges and defines a non-zero holomorphic function.

Remark. The name of this form comes from the theory of elliptic curves. It corresponds to the discriminant of the curve $\mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$. Elliptic curves are closely related to modular forms but we will not discuss that theory here.

Proposition 3.12. *The function Δ is a cusp form of weight 12 and group $\Gamma(1)$ equal to $\frac{1}{1728}(E_4^3 - E_6^2)$.*

Proof (sketch). For the proof that Δ is a modular form of weight 12, see [12]. It relies on derivative of modular forms. A subject we will not discuss here. Looking at the definition of Δ, it is obvious that it is a cusp form. From theorem 3.9, we know that it must be a linear combination of E_4^3 and E_6^2. Since it is a cusp form, Δ is equal to a constant times $E_4^3 - E_6^2$. Looking at the first non-zero Fourier coefficient of Δ, we conclude that this constant is $\frac{1}{1728}$. □

3.3 Theta series

Definition 3.13. The *Thetanullwerte* are the functions:

$$\theta_{00}(z) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2 z}$$
$$\theta_{01}(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{i\pi n^2 z}$$
$$\theta_{10}(z) = \sum_{n\in\mathbb{Z}} e^{i\pi (n+\frac{1}{2})^2 z}$$

These functions come from the *Jacobi theta function*

$$\vartheta : \mathbb{C}\times\mathbb{H} \to \mathbb{C},\ (z,\tau)\mapsto \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau + 2i\pi n z}.$$

Since $\mathrm{Im}(\tau) > 0$, the series converges. $\vartheta(z,\tau)$ appears in various forms in a lot of different contexts. A quite extensive exposition of this function is given in [16]. For example, it gives the unique solution to the one-dimensional heat equation $\frac{\partial\vartheta}{\partial\tau} = \frac{1}{4\pi}\frac{\partial^2\vartheta}{\partial z^2}$ with a periodic initial data. It also comes naturally as invariant function under the two following actions on complex maps f (for a fixed $\tau\in\mathbb{H}$):

$$(S_b f)(z) := f(z+b) \qquad\qquad (T_a f)(z) := e^{i\pi a^2\tau + 2i\pi a z} f(z + a\tau).$$

Our functions are the special values

$$\theta_{00}(\tau) = \vartheta(0,\tau)$$
$$\theta_{01}(\tau) = (S_{\frac{1}{2}}\vartheta)(0,\tau) = \vartheta(\tfrac{1}{2},\tau)$$
$$\theta_{10}(\tau) = (T_{\frac{1}{2}}\vartheta)(0,\tau) = e^{i\pi\tau/4}\vartheta(\tfrac{1}{2}\tau,\tau)$$

We also have the function $\theta_{11}(\tau) = (S_{\frac{1}{2}}T_{\frac{1}{2}}\vartheta)(0,\tau)$, but it is null everywhere:

$$\theta_{11}(\tau) = e^{i\pi\tau/4 + i\pi/2}\vartheta(\tfrac{1}{2}\tau + \tfrac{1}{2},\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau/4+i\pi/2} e^{i\pi n^2\tau + i\pi n(\tau+1)}$$
$$= \sum_{n\in\mathbb{Z}} e^{i\pi\tau(n^2+n+\frac{1}{4})} e^{i\pi(n+\frac{1}{2})} = i\sum_{n\in\mathbb{Z}} (-1)^n e^{i\pi\tau(n+\frac{1}{2})^2} = 0.$$

The sum vanishes because the $n+\frac{1}{2}$ term cancel with the $-(n+1)+\frac{1}{2}$ one. Our theta functions satisfy the following modular relations:

$$\theta_{00}^4(z+1) = \theta_{01}^4(z) \qquad \theta_{01}^4(z+1) = \theta_{00}^4(z) \qquad \theta_{10}^4(z+1) = -\theta_{10}^4(z)$$

and

$$z^{-2}\theta_{00}^4\left(-\frac{1}{z}\right) = -\theta_{00}^4(z) \qquad z^{-2}\theta_{01}^4\left(-\frac{1}{z}\right) = -\theta_{10}^4(z) \qquad z^{-2}\theta_{10}^4\left(-\frac{1}{z}\right) = -\theta_{01}^4(z)$$

So these functions behave like a modular form of $\frac{1}{2}$ weight would. In particular, θ_{00}^4, θ_{01}^4 and θ_{10}^4 are modular forms of weight 2 and group $\Gamma(2)$. The first three relations are easy to prove and the last three come from the following general transformation formula.

Lemma 3.14. *Let* $\theta_a(z) := \sum_{n\in\mathbb{Z}} e^{i\pi(n+a)^2 z}$*, for* $a \in \mathbb{R}$*. Then*

$$\theta_a(z) = \frac{1}{\sqrt{iz}} \sum_{n\in\mathbb{Z}} e^{-i\pi n^2/z} e^{2i\pi a n}$$

Proof. We denote $\tilde{\theta}_a(z) := \sum_{n\in\mathbb{Z}} e^{\pi(n+a)^2 z}$, so $\tilde{\theta}_a(iz) = \theta_a(z)$. Using Poisson summation formula (theorem 2.7) for the lattice $\mathbb{Z} \subseteq \mathbb{R}$ and the shift $y = 0$, we get

$$\tilde{\theta}_a(z) = \sum_{n\in\mathbb{Z}} e^{\pi(n+a)^2 z} = \sum_{n\in\mathbb{Z}} \mathcal{F}_n\left(e^{\pi(n+a)^2 z}\right),$$

where the Fourier transform is on the variable n. In the next section, we will see in lemma 3.19 that this Fourier transform is in fact

$$\mathcal{F}_n\left(e^{\pi(n+a)^2 z}\right) = z^{-1/2} e^{\pi n^2/z} e^{2i\pi a n}.$$

We also used the classical rule $\mathcal{F}_n(f(n+n_0))(m) = e^{2i\pi n_0 m} f(m)$. We conclude that

$$\tilde{\theta}_a(z) = \frac{1}{\sqrt{z}} \sum_{n\in\mathbb{Z}} e^{\pi n^2/z} e^{2i\pi a n}.$$

Transposing to the imaginary axis via $z \mapsto iz$, we get

$$\theta_a(z) = \tilde{\theta}_a(iz) = \frac{1}{\sqrt{iz}} \sum_{n\in\mathbb{Z}} e^{-i\pi n^2/z} e^{2i\pi a n}.$$

□

We get the transformation formula for θ_{00} by taking $a = 0$ and for θ_{01} and θ_{10} by taking $a = \frac{1}{2}$ and using that $z \mapsto -\frac{1}{z}$ is its own inverse.

Proposition 3.15 (Jacobi identity)**.** *The Jacobi theta functions satisfy*

$$\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4.$$

Proof. The proof relies on the following identity of quadratic forms:

$$(n_1+n_2+n_3+n_4)^2+(n_1+n_2-n_3-n_4)^2+(n_1-n_2+n_3-n_4)^2+(n_1-n_2-n_3+n_4)^2 = 4(n_1^2+n_2^2+n_3^2+n_4^2).$$

We will prove

$$\theta_{00}^4 + \theta_{01}^4 + \theta_{10}^4 + \theta_{11}^4 = 2\theta_{00}^4.$$

Since θ_{11} is zero everywhere, this is equivalent to the proposition. The left-hand side is:

$$\theta_{00}^4(z) + \theta_{01}^4(z) + \theta_{10}^4(z) + \theta_{11}^4(z) = \sum_{n_1,n_2,n_3,n_4\in\mathbb{Z}} \Big(e^{i\pi z \Sigma n_i^2} + (-1)^{\Sigma n_i} e^{i\pi z \Sigma n_i^2} + e^{i\pi z \Sigma (n_i+\frac{1}{2})^2}$$
$$+(-1)^{\Sigma n_i} e^{i\pi z \Sigma (n_i+\frac{1}{2})^2}\Big) = \sum_{\substack{(n_1,n_2,n_3,n_4)\in\mathbb{Z}^4\cup(\mathbb{Z}+\frac{1}{2})^4 \\ n_1+n_2+n_3+n_4\equiv 0 \mod 2}} 2e^{i\pi z \Sigma n_i^2}.$$

The sums in the exponentials always go from $i = 1$ to 4. The first equality is obtained by expanding each power and the second one by canceling the terms and regrouping everything in one sum. Let

$$\begin{aligned} m_1 &= \frac{1}{2}(n_1 + n_2 + n_3 + n_4), \\ m_2 &= \frac{1}{2}(n_1 + n_2 - n_3 - n_4), \\ m_3 &= \frac{1}{2}(n_1 - n_2 + n_3 - n_4), \\ m_4 &= \frac{1}{2}(n_1 - n_2 - n_3 + n_4). \end{aligned}$$

The identity above implies that

$$m_1^2 + m_2^2 + m_3^2 + m_4^2 = n_1^2 + n_2^2 + n_3^2 + n_4^2.$$

Moreover, in the sum, $n_1 + n_2 + n_3 + n_4 \equiv 0 \pmod 2$ and the n_i are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$. It implies that the m_i are all in $\mathbb{Z}$. One can easily compute that it is even a bijection to $\mathbb{Z}^4$. Hence, we conclude that:

$$\theta_{00}^4(z) + \theta_{01}^4(z) + \theta_{10}^4(z) + \theta_{11}^4(z) = \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}} 2e^{i\pi z \Sigma m_i} = 2\theta_{00}^4(z).$$

□

Proposition 3.16. *The relation between the theta functions and the modular discriminant is*

$$256\Delta = \theta_{00}^8 \theta_{01}^8 \theta_{10}^8.$$

Proof. Up to a constant, Δ is the only cusp form of weight 12 on $\Gamma(1)$. Hence, one just need to see that the function on the right is also a cusp form for $\Gamma(1)$ and compare the Fourier coefficients. □

Theorem 3.17. *θ_{01} and θ_{10} forms a basis of the modular forms over $\Gamma(2)$, i.e. $\mathcal{M}(\Gamma(2)) \cong \mathbb{C}[\theta_{01}^4, \theta_{10}^4]$.*

Proof. See [9], section 2.1.2. □

3.4 Laplace transform

The Laplace transform is an integral operator on functions, similar to the Fourier transform. One advantage of this operator in regards to the Fourier transform is that it takes initial conditions into account. Here, our interest is that it can be seen as a continuous combination of eigenfunctions of the Fourier transform. Hence, it will be useful for our constructions of the magic functions as a sum of two eigenfunctions of the Fourier transform, of eigenvalues $+1$ and -1.

Definition 3.18. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ an L^1 function. The *Laplace transform* of f is

$$\mathcal{L}(f)(s) := \int_0^\infty f(t) e^{-st} dt.$$

We will consider the evaluation of Laplace transform at $s = \pi |x|^2$, for some $x \in \mathbb{R}^n$. In one dimension, it relates to the function $e^{-\pi x^2}$ which is a eigenfunction of the Fourier transform.

Lemma 3.19. *The Gaussian $e^{-\pi x^2}$ is an eigenfunction of the Fourier transform of eigenvalue 1:*

$$\mathcal{F}\left(e^{-\pi x^2}\right)(y) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2i\pi xy} dx = e^{-\pi y^2}$$

More generally, for $x \in \mathbb{R}^n$ and $a \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$, the n-dimension Fourier transform of $e^{-a\pi|x|^2}$ is given by:

$$\mathcal{F}\left(e^{-a\pi|x|^2}\right)(y) = \int_{\mathbb{R}^n} e^{-a\pi|x|^2} e^{-2i\pi\langle x,y\rangle} dx = a^{-n/2} e^{-\pi|y|^2/a},$$

for all $y \in \mathbb{R}^n$, where $a^{1/2}$ is the only square root of a with real part greater than 0.

Proof. We denote $g(y) := \mathcal{F}\left(e^{-a\pi|x|^2}\right)(y)$. If we differentiate g with respect to y_1, we get

$$\frac{\partial g}{\partial y_1}(y) = -2i\pi \int_{\mathbb{R}^n} x_1 e^{-a\pi|x|^2} e^{-2i\pi\langle x,y\rangle} dx.$$

Integrating by parts with respect to x_1, we obtain

$$\begin{aligned}\frac{\partial g}{\partial y_1}(y) &= \int_{\mathbb{R}^{n-1}} \left[\left(\frac{i}{a} e^{-a\pi|x|^2}\right) e^{-2i\pi\langle x,y\rangle}\right]_{x_1=-\infty}^{\infty} dx_2 \dots dx_n - \int_{\mathbb{R}^n} \left(\frac{i}{a} e^{-a\pi|x|^2}\right)\left(-2i\pi y_1 e^{-2i\pi\langle x,y\rangle}\right) dx \\ &= -\frac{2\pi y_1}{a} \int_{\mathbb{R}^n} e^{-a\pi|x|^2} y_1 e^{-2i\pi\langle x,y\rangle} dx = -\frac{2\pi y_1}{a} g(y).\end{aligned}$$

This gives a one-dimensional differential equation that one easily solves to get

$$g(y) = C(y_2, \dots, y_n) e^{-\pi y_1^2/a},$$

where the constant depends on all the other coordinates $y_2, \dots, y_n$. Doing the same computation for each coordinate y_i, $i = 1, \dots, n$, we conclude that

$$g(y) = c e^{-\pi|y|^2/a}.$$

Finally, we just have to evaluate g to find the value of the constant. We begin in one dimension. We get

$$g(0) = \int_{-\infty}^{\infty} e^{-a\pi x^2} dx = \int_{-a\infty}^{a\infty} \frac{1}{\sqrt{a}} e^{-\pi x^2} dx,$$

We made the change of variable $x \mapsto \sqrt{a}x$, with the complex square root such that $\mathrm{Re}(\sqrt{a}) > 0$. Writing the integral with limits and using the holomorphy of the function, we have

$$\begin{aligned}\int_{-a\infty}^{a\infty} \frac{1}{\sqrt{a}} e^{-\pi x^2} dx &= \lim_{r\to\infty} \int_{-ar}^{ar} \frac{1}{\sqrt{a}} e^{-\pi x^2} dx \\ &= \lim_{r\to\infty} \left(\int_{-ar}^{-r} \frac{1}{\sqrt{a}} e^{-\pi x^2} dx + \int_{-r}^{r} \frac{1}{\sqrt{a}} e^{-\pi x^2} dx + \int_{r}^{ar} \frac{1}{\sqrt{a}} e^{-\pi x^2} dx\right).\end{aligned}$$

The first and the last integrals are bounded by $r|a|^{1/2} e^{-\pi r^2}$ and therefore converge to 0. We conclude that

$$g(0) = \int_{-\infty}^{\infty} e^{-a\pi x^2} dx = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-\pi x^2} dx = \frac{1}{\sqrt{a}},$$

by using the classical result that $\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1$.

We can now compute the constant c:

$$c = g(0) = \int_{\mathbb{R}^n} e^{-a\pi|x|^2} dx = \left(\int_{-\infty}^{\infty} e^{-a\pi y} dy \right)^n = a^{-n/2},$$

□

If f is sufficiently well behaved (like an admissible or a Schwartz function), then one can compute the Fourier transform of the function

$$g(x) := \mathcal{L}(f)(\pi|x|^2) = \int_0^\infty f(t) e^{-\pi|x|^2 t} dt$$

by exchanging integrals. For $x \in \mathbb{R}^n$, the result is

$$\hat{g}(x) = \int_0^\infty f(t) t^{-n/2} e^{-\pi|x|^2/t} dt = \int_0^\infty f\left(\frac{1}{t}\right) t^{n/2-2} e^{-\pi|x|^2 t} dt,$$

where we made the change of variable $t \mapsto \frac{1}{t}$ in the last equality. We see that taking the Fourier transform is the same as replacing $f(t)$ by $t^{n/2-2} f(1/t)$. This looks like the transformation formula of some modular form. In particular, if $f(t) = \phi(it)$, where ϕ is a modular form of weight $k = 2 - n/2$, we have that $f(1/t) = \phi(i/t) = \phi(-1/it) = (it)^k \phi(it) = (it)^k f(t)$, hence $\hat{g} = i^{2-n/2} g$, i.e. g is an eigenfunction of the Fourier transform. This is how we use modular forms and Laplace transform to construct the magic functions. For now, this approach seems to have two problems. First, this only gives a function of eigenvalue i^k, here of eigenvalue -1 in both cases. This is not a big issue since there is a lot of variants of the classical definition of modular forms and one can construct eigenfunctions with other eigenvalues. The second problem is a more serious obstacle. We don't have any control on the zeros of the eigenfunction constructed using Laplace transform. To deal with this, Viazovska inserted by brute force the desired roots with a sine factor in front of the Laplace transform. As we will see below, both eigenfunctions in dimensions 8 and 24 have the form

$$\sin(\pi|x|^2/2)^2 \int_0^\infty f(t) e^{-\pi|x|^2 t} dt.$$

One big difficulty is then to control the behavior of this function under the Fourier transform. We have to understand which conditions must satisfy ϕ under inversion to be an eigenfunction.

4 The sphere packing problem in dimension 8

In this chapter, we construct the two Fourier eigenfunctions with zeros at all non-zero lattice points and combine them to get a tight bound for the density of a sphere packing in 8 dimensions. We first detail the construction for the positive eigenfunction. We derive from this construction sufficient conditions on the function to have all the right properties. These conditions are mostly valid in 24 and other dimensions divisible by 8. Then we use these conditions to find the first eigenfunction inside a finite dimensional space. In the second section, we adapt the conditions to get another eigenfunction with -1 eigenvalue. We conclude by creating a function combining these two eigenfunction that satisfies the conditions of theorems 2.8 and 2.9 and that gives the right bound on the sphere packing density.

4.1 The +1 Fourier eigenfunction

We begin by the construction of a +1 Fourier eigenfunction in 8 dimensions. To simplify the computations, we define our functions such that they take imaginary values. Following the ideas of the introduction and the remark below theorem 2.8, we look for a function satisfying the following properties:

1. It has double zeros at all vector length $\sqrt{2k}$ for $k \geq 2$, a simple zero at $|x| = \sqrt{2}$ and does not vanish at 0.

2. It takes purely imaginary values.

3. Its 8-dimensional Fourier transform is itself.

4. It is a radial Schwartz function.

We do the computations mostly in a general setting, to be able to apply the same results to the other cases. Our reduction on dimension is only that 8 divides n. This condition is mainly useful to simplify $i^{n/2} = 1$. What follows can probably also be done for $n \equiv 4 \pmod 8$ or even $n \equiv 2, 6 \pmod 8$.

We take a slightly different function than the one proposed in the end of last chapter. We suppose our function is of the form

$$a(x) = \sin(\pi|x|^2/2)^2 \int_0^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt, \tag{4.1}$$

where $k = 2 - n/2$ and ϕ is a function satisfying some modular properties related to the weight k. We will see that in fact ϕ is a quasimodular form of weight $k+2$ and depth 2 on $\mathrm{SL}(2,\mathbb{Z})$. Hence, evaluating ϕ at $-1/t$ instead of t and adding a t^{-k} term gives factors in t and t^2, as explained in the remark after corollary 3.10. We will see that these factors cancel the zeros in excess and simplify the Fourier transform of $a(x)$. Also, ϕ is 1-periodic.

To get some information on $a(x)$, we will rewrite this function in two other forms. The second form is an expansion of ϕ when $\mathrm{Im}(t) \to \infty$. The desired location of the zeros gives us a condition on this expansion. This will imply that ϕ should be a quasimodular form of depth at least 1. This form also gives a condition to ensure that we have a purely imaginary valued function. To get the third form, we expand the sine and deform the integration paths. Our requirement that $a(x)$ is a Fourier eigenfunction implies that the depth of ϕ is 2. This form also allows us to compute that $a(x)$ is a Schwartz function at the price of a condition on the Fourier expansion of ϕ. Finally, these two forms are holomorphic as a function of $r = |x|$ and equal in a neighborhood of $+\infty$. Therefore, they are equal everywhere by analytic continuation.

In the end of this section, we compute the function for $n = 8$ using the conditions we found on ϕ. A computer algebra system will be required for these calculations. We get a function of the form $\phi = \frac{\Phi}{\Delta}$, where Φ is a quasimodular form of weight 12 and Δ is the modular discriminant.

4.1.1 The zeros and the imaginary values

We first look at the location of the zeros of $a(x)$ to get information on its Fourier series. Here, we go back to the real notations for the Laplace transform, i.e.

$$a(x) = -i \sin(\pi|x|^2/2)^2 \int_0^{\infty} \phi\left(\frac{i}{t}\right) t^{-k} e^{-\pi|x|^2 t} dt$$

The squared sine gives double zeros at each lattice point and a zero of degree 4 at 0, because of the $|x|^2$ term inside. So what we would like is a pole of degree four at 0 and a pole of degree one at $|x| = \sqrt{2}$ for the Laplace transform of ϕ. Since ϕ is 1-periodic, it admits a Fourier series. If ϕ is a quasimodular form, $\phi(i/t)$ can be written as a series in terms of the form $t^j e^{-2\pi t m}$. Therefore, we begin by computing Laplace transforms corresponding to this. Let $m \in \mathbb{Z}$, $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$ such that $|x|^2 > -2m$. We have

$$\mathcal{L}(e^{-2\pi t m})(\pi|x|^2) = \int_0^\infty e^{-\pi t(2m+|x|^2)} dt = -\frac{1}{\pi(2m+|x|^2)} e^{-\pi t(2m+|x|^2)} \Big|_0^\infty = \frac{1}{\pi(2m+|x|^2)},$$

$$\mathcal{L}(t^j e^{-2\pi t m})(\pi|x|^2) = \int_0^\infty t^j e^{-\pi t(2m+|x|^2)} dt = -\frac{t^j}{\pi(2m+|x|^2)} e^{-\pi t(2m+|x|^2)} \Big|_0^\infty + \frac{j}{\pi(2m+|x|^2)} \int_0^\infty t^{j-1} e^{-\pi t(2m+|x|^2)} dt = \frac{j!}{(\pi(2m+|x|^2))^{j+1}},$$

where the last equality follows by induction on k. To cancel the zeros of the sine, we want a term in $e^{2\pi t}$ to have a simple pole for $|x| = \sqrt{2}$ and a term in t to have a pole of degree 4 for $x = 0$. This gives an expansion of ϕ of the form

$$\phi\left(\frac{i}{t}\right) t^{-k} = c e^{2\pi t} + (dt + f) + O(t^m e^{-2\pi t}), \tag{4.2}$$

as $t \to \infty$, for c and d non-zero constants. All of this is only valid for $|x| > \sqrt{2}$, because our minimal Fourier coefficient is $m = -1$. A way to get the expansion above is that ϕ is a quasimodular form of depth at least 1. We rewrite the function as

$$a(x) = -\, i \sin(\pi|x|^2/2)^2 \left(\frac{c}{\pi(|x|^2-2)} + \frac{d}{\pi^2|x|^4} + \frac{f}{\pi|x|^2} + \int_0^\infty \left(\phi\left(\frac{i}{t}\right) t^{-k} - (c e^{2\pi t} + dt + f) \right) e^{-\pi|x|^2 t} dt \right). \tag{4.3}$$

This is the second form of $a(x)$ we consider. It is obtained by integrating the expansion 4.2 term by term. Now, the integral converges everywhere and the function has the right zeros. Moreover, we see that if ϕ take real values on the imaginary line, the values of $a(x)$ are purely imaginary. This is the case for instance if the Fourier coefficients of ϕ are all real. With this condition, we will ensure us to create a real-valued function in the end. Moreover, considering $a(r)$ as a 1-dimensional function of $r = |x|$, one can compute all its derivatives by exchanging the limit with the integral if it is allowed. This gives an integral in $a'(r)$ of the form

$$-r\pi \int_0^\infty \phi\left(\frac{1}{t}\right) t^{-k+1} e^{-\pi r^2 t} dt,$$

where $r = |x|$ and we took a radial derivative. $a(r)$ is analytic if the integral converges. This is the case in equation 4.3 with our modified Fourier expansion. Hence, the first form 4.1 converges only for $r > \sqrt{2}$ and the second form is an analytic continuation of it to $r \geq 0$.

4.1.2 The Fourier transform

Now, we want $a(x)$ to be an eigenfunction of the Fourier transform, using the modular properties of ϕ. One can directly compute the Fourier transform of $a(x)$, just by expanding the sine as a sum of exponentials and inserting the Fourier transform inside the integrals. This does not conclude, mainly because the bounds of the integrals change during the computation. Instead of doing this, we deform our paths of integration so that they all go through i and use the invariance of i under the inversion $t \mapsto -\frac{1}{t}$.

First, developing the sine we get

$$\sin(\pi|x|^2/2)^2 = -\frac{1}{4}\left(e^{i\pi|x|^2/2} - e^{-i\pi|x|^2/2}\right)^2 = \frac{1}{4}\left(2 - e^{i\pi|x|^2} - e^{-i\pi|x|^2}\right).$$

Hence, the function $a(x)$ can be written as the sum

$$\begin{aligned}4a(x) &= 2\int_0^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt - \int_0^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2(t+1)} dt - \int_0^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2(t-1)} dt \\ &= 2\int_0^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt - \int_1^{i\infty+1} \phi\left(\frac{-1}{t-1}\right) (t-1)^{-k} e^{i\pi|x|^2 t} dt \\ &- \int_{-1}^{i\infty-1} \phi\left(\frac{-1}{t+1}\right) (t+1)^{-k} e^{i\pi|x|^2 t} dt.\end{aligned}$$

We deform the integration paths to make them all pass through i. We use equation 4.2 to see that the integral vanishes at infinity if $|x|$ is big enough (to be precise, $|x| > \sqrt{2}$).

$$\begin{aligned}4a(x) &= 2\int_0^{i} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt + 2\int_i^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt \\ &- \int_1^{i} \phi\left(\frac{-1}{t-1}\right) (t-1)^{-k} e^{i\pi|x|^2 t} dt - \int_i^{i\infty} \phi\left(\frac{-1}{t-1}\right) (t-1)^{-k} e^{i\pi|x|^2 t} dt \\ &- \int_{-1}^{i} \phi\left(\frac{-1}{t+1}\right) (t+1)^{-k} e^{i\pi|x|^2} dt - \int_i^{i\infty} \phi\left(\frac{-1}{t+1}\right) (t+1)^{-k} e^{i\pi|x|^2} dt \\ &= 2\int_0^{i} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt - \int_1^{i} \phi\left(\frac{-1}{t-1}\right) (t-1)^{-k} e^{i\pi|x|^2 t} dt - \int_{-1}^{i} \phi\left(\frac{-1}{t+1}\right) (t+1)^{-k} e^{i\pi|x|^2 t} dt \\ &+ \int_i^{i\infty} \left(2\phi\left(\frac{-1}{t}\right) t^{-k} - \phi\left(\frac{-1}{t-1}\right) (t-1)^{-k} - \phi\left(\frac{-1}{t+1}\right) (t+1)^{-k}\right) e^{i\pi|x|^2 t} dt\end{aligned}$$

We know from lemma 3.19 and the discussion at the end of last chapter that the Fourier transform gives a change of variable $t \mapsto -\frac{1}{t}$. Hence, if all goes well, it will exchange the two middle integrals and the first and the last one. This gives a condition on $\tilde{\phi}(t) := 2\phi\left(-\frac{1}{t}\right) t^{-k} - \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k} - \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k}$.

Proposition 4.1. *Suppose that $\tilde{\phi} = -2\phi$. Then $a(x)$ is an eigenfunction of the Fourier transform.*

Proof. Lemma 3.19 tells us that the Fourier transform of the first integral of $4a(x)$ is

$$\mathcal{F}\left(\int_0^i \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt\right) = \int_0^i \phi\left(\frac{-1}{t}\right) t^{-k} (-it)^{-n/2} e^{i\pi|x|^2\left(\frac{-1}{t}\right)} dt = -\int_i^{i\infty} \phi(t)\, e^{i\pi|x|^2 t} dt.$$

In the last equality, we made the change of variable $t \mapsto -\frac{1}{t}$ and used that 8 divides n to simplify $(-i)^{n/2}$. Similarly, the Fourier transforms of the other integrals are

$$\begin{aligned}\mathcal{F}\left(\int_1^i \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k}e^{i\pi|x|^2t}dt\right) &= \int_1^i \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k}t^{-n/2}e^{i\pi|x|^2\left(\frac{-1}{t}\right)}dt \\ &= \int_{-1}^i \phi\left(\frac{-1}{-\frac{1}{t}-1}\right)\left(-\frac{1}{t}-1\right)^{-k}t^{-k}e^{i\pi|x|^2t}dt \\ &= \int_{-1}^i \phi\left(1-\frac{1}{t+1}\right)(t+1)^{-k}e^{i\pi|x|^2t}dt \\ &= \int_{-1}^i \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k}e^{i\pi|x|^2t}dt\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left(\int_{-1}^i \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k}e^{i\pi|x|^2t}dt\right) &= \int_1^i \phi\left(\frac{-1}{-\frac{1}{t}+1}\right)\left(-\frac{1}{t}+1\right)^{-k}t^{-k}e^{i\pi|x|^2t}dt \\ &= \int_1^i \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k}e^{i\pi|x|^2t}dt\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left(\int_i^{i\infty} \tilde{\phi}(t)e^{i\pi|x|^2t}dt\right) &= \int_i^{i\infty} \tilde{\phi}(t)t^{-n/2}e^{i\pi|x|^2\left(\frac{-1}{t}\right)}dt \\ &= -\int_0^i \tilde{\phi}\left(\frac{-1}{t}\right)t^{-k}e^{i\pi|x|^2t}dt\end{aligned}$$

We used that ϕ is 1-periodic and that k is even. Using $\tilde{\phi}(t) = -2\phi(t)$, the last integral is equal to

$$2\int_0^i \phi\left(\frac{-1}{t}\right)t^{-k}e^{i\pi|x|^2t}dt.$$

Combining all together, we see that $a(x)$ is a Fourier eigenfunction. □

Proposition 4.1 supposes that $-2\phi(t) = 2\phi\left(-\frac{1}{t}\right)t^{-k} - \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k} - \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k}$. This can be simplified. We saw that ϕ has to be a quasimodular form of depth at least 1 and weight $k+2$. If it is actually of depth 2, we have, according to corollary 3.10, that

$$\phi(-1/t)t^{-k} = \phi(t)t^2 + \phi_1(t)t + \phi_2(t),$$

where ϕ_1 and ϕ_2 are 1-periodic function. Then:

$$\begin{aligned}\tilde{\phi}(t) &= 2\phi\left(-\frac{1}{t}\right)t^{-k} - \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k} - \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k} \\ &= 2\phi(t)t^2 - \phi(t-1)(t-1)^2 - \phi(t+1)(t+1)^2 \\ &+ 2\phi_1(t)t - \phi_1(t-1)(t-1) - \phi_1(t+1)(t+1) \\ &+ 2\phi_2(t) - \phi_2(t+1) - \phi_2(t-1) \\ &= -2\phi(t).\end{aligned}$$

In conclusion, the third form of $a(x)$ that we consider is

$$4a(x) = 2\int_0^i \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt - \int_1^i \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k} e^{i\pi|x|^2 t} dt \tag{4.4}$$
$$- \int_{-1}^i \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k} e^{i\pi|x|^2 t} dt - 2\int_i^{i\infty} \phi(t) e^{i\pi|x|^2 t} dt.$$

4.1.3 The Schwartz function

The last thing we have to check is that our function is admissible. We even prove that it is a Schwartz function up to some condition. ϕ is 1-periodic so it has a Fourier series. We first suppose that this series does not have infinitely many coefficients for negative m. Therefore, it is of the form $\phi(t) = \sum_{m=m_0}^{\infty} f_m q^m$, where $q = e^{2i\pi t}$ as usual. Since ϕ is a quasimodular form, the bound of theorem 3.5 applies, i.e. $|f_m| \leq C_1 e^{C_2\sqrt{|m_0 m|}}$ with C_1, C_2 positive constants. So if $\mathrm{Im}(t) > 1/2$, we have the bound

$$\begin{aligned}
|\phi(t)| &= \left|\sum_{m=m_0}^{\infty} f_m q^m\right| \leq \left|\sum_{m=m_0}^{(C_2+1)^2|m_0|-1} f_m q^m\right| + \left|\sum_{m=(C_2+1)^2|m_0|}^{\infty} f_m q^m\right| \\
&\leq e^{-2\pi m_0} \sum_{m=m_0}^{(C_2+1)^2|m_0|-1} |f_m| e^{-2\pi(m-m_0)\,\mathrm{Im}(t)} + C_1 \sum_{m=(C_2+1)^2|m_0|}^{\infty} e^{C_2\sqrt{|m_0|m}} e^{-2\pi m\,\mathrm{Im}(t)} \\
&\leq e^{-2\pi m_0} \sum_{m=m_0}^{(C_2+1)^2|m_0|-1} |f_m| + C_1 \sum_{m=(C_2+1)^2|m_0|}^{\infty} e^{C_2\sqrt{\frac{m^2}{(C_2+1)^2}}} e^{-2\pi\,\mathrm{Im}(t)\left(\frac{C_2^2+2C_2}{(C_2+1)^2}m + |m_0|\right)} \\
&\leq C_3 e^{-2\pi m_0} + C_1 e^{-2\pi|m_0|\,\mathrm{Im}(t)} \sum_{m=(C_2+1)^2|m_0|}^{\infty} e^{\frac{C_2}{C_2+1}m} e^{-\pi\frac{C_2}{C_2+1}m} e^{-\pi\frac{C_2}{(C_2+1)^2}m} \\
&\leq C e^{-2\pi m_0\,\mathrm{Im}(t)},
\end{aligned}$$

with the positive constant C depending on m_0. In the third line, we used that $m \geq (C_2+1)^2|m_0|$ in the second sum. The last inequality used that the sum on the fourth line converges.

Now, we approximate the first integral in the third form (equation 4.4) of $a(x)$:

$$\left|\int_0^i \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt\right| = \left|\int_{i\infty}^i \phi(t) t^{-n/2} e^{-i\pi|x|^2/t} dt\right| \leq C\int_1^{\infty} e^{-2\pi m_0 t} e^{-\pi|x|^2/t} dt.$$

We see that we need at least $m_0 \geq 1$ to have the convergence of the integral at ∞. This gives another condition on ϕ. For the second and the third integrals, we have the same kind of computation:

$$\left|\int_1^i \phi\left(\frac{-1}{t-1}\right)(t-1)^{-k} e^{i\pi|x|^2 t} dt\right| = \left|\int_{i\infty}^{\frac{-1}{i-1}} \phi(t) t^{-n/2} e^{i\pi|x|^2\left(\frac{-1}{t}+1\right)} dt\right| \leq C\int_{1/2}^{\infty} e^{-2\pi m_0 t} e^{-\pi|x|^2/t} dt$$

$$\left|\int_{-1}^i \phi\left(\frac{-1}{t+1}\right)(t+1)^{-k} e^{i\pi|x|^2 t} dt\right| = \left|\int_{i\infty}^{\frac{-1}{i+1}} \phi(t) t^{-n/2} e^{i\pi|x|^2\left(\frac{-1}{t}-1\right)} dt\right| \leq C\int_{1/2}^{\infty} e^{-2\pi m_0 t} e^{-\pi|x|^2/t} dt$$

We can give the same bound to these three integrals. To do this, we can compare it to a K-Bessel function (as in Viazovska's paper) or more elementary, we can split it in two parts:

$$\begin{aligned}\int_{1/2}^{\infty} e^{-2\pi m_0 t} e^{-\pi|x|^2/t} dt &\leq \int_{1/2}^{|x|} e^{-2\pi t} e^{-\pi|x|^2/t} dt + \int_{|x|}^{\infty} e^{-2\pi t} e^{-\pi|x|^2/t} dt \\ &\leq e^{-\pi|x|} \int_{1/2}^{|x|} e^{-2\pi t} dt + |x|^2 \int_{|x|^{-1}}^{\infty} e^{-2\pi|x|^2 t} e^{-\pi/t} dt \\ &\leq C' e^{-\pi|x|} + |x|^2 \int_{|x|^{-1}}^{\infty} e^{-2\pi|x|^2 t} dt \leq C' e^{-\pi|x|} - \frac{1}{2\pi} e^{-2\pi|x|^2 t}\Big|_{|x|^{-1}}^{\infty} \\ &\leq C'' e^{-\pi|x|}\end{aligned}$$

For the last integral, the computation is a bit different but easier:

$$\left|\int_{i}^{i\infty} \phi(t) e^{i\pi|x|^2 t} dt\right| \leq C \int_{1}^{\infty} e^{-2\pi m_0 t} e^{-\pi|x|^2 t} dt \leq C \frac{e^{-\pi(2m_0+|x|^2)}}{\pi(2m_0+|x|^2)} \leq C_\infty \frac{e^{-\pi|x|^2}}{|x|^2}.$$

Hence, there exist constants C'' and C_∞ such that

$$|a(x)| \leq C'' e^{-\pi|x|} + C_\infty \frac{e^{-\pi|x|^2}}{|x|^2}.$$

So $a(x)$ decrease faster at infinity that any inverse power of polynomials. And this is also true for any derivative of $a(r)$, as a function of $r = |x|$, because each derivation just add a polynomial factor in the integral. So integrating by parts returns us to the case before, up to a rational function in $|x|$. Therefore, $a(r)$ is a Schwartz function. These bounds also prove that the form 4.4 converges for all $x \in \mathbb{R}^n$. Hence, it is even an analytic function as a function of r. By analytic continuation, it is equal to the form 4.3.

We got all the conditions we need. Note again that everything we did is independent of the dimension (as long as 8 divides n). Here is the summary of what is above:

Theorem 4.2. *Let n a multiple of 8, $k = 2 - n/2$ and ϕ a quasimodular form of depth 2 and weight $k+2$ on $\Gamma(1)$. If ϕ satisfies:*

1. *ϕ grows like $\phi(i/t)t^{-k} = ce^{2\pi t} + (dt + f) + O(t^2 e^{-2\pi t})$ as $t \to \infty$, with $c, d \neq 0$.*
2. *ϕ has real Fourier coefficients.*
3. *The Fourier series of ϕ is $\phi(t) = \sum_{m=1}^{\infty} f_m q^m$.*

Then $a(x)$ is a $+1$ Fourier eigenfunction taking values in $i\mathbb{R}$ and a radial Schwartz function with a simple zero at $|x| = \sqrt{2}$ and double zeros at $|x| = \sqrt{2k}$ for all $k \geq 2$.

4.1.4 Computation of the function

We begin now the explicit search for the $+1$ eigenfunction. For $n = 8$, we have that $k + 2 = 0$. We don't have any holomorphic quasimodular form of this weight on $\mathrm{SL}(2, \mathbb{Z})$. It's possible to search our function is the more general space of weakly holomorphic quasimodular forms, but this space has infinite dimensions. To reduce us to a better space, we notice that $\Phi := \Delta\phi$ is a quasimodular form of weight 12. Corollary 3.10 tells us that it belongs to the space $\mathbb{C}[E_2, E_4, E_6]$. This space has 4 dimensions and is generated by $E_2^2 E_4^2$, $E_2 E_4 E_6$, E_4^3 and E_6^2. The theorem 4.2 can be rewritten for Φ as:

Proposition 4.3. *Let Φ is a quasimodular form of weight 12 and depth 2 on $\Gamma(1)$ such that:*

1. *$\Phi(i/t)t^{-10} = c + (dt+f)e^{-2\pi t} + O(t^2 e^{-4\pi t})$ as $t \to \infty$, with $c, d \neq 0$.*
2. *The Fourier coefficients of Φ are all reals.*
3. *The Fourier series of Φ is $\Phi(t) = \sum_{m=2}^{\infty} f_m e^{2i\pi m}$.*

Then $\phi = \Phi/\Delta$ satisfies the conditions of theorem 4.2 for $n = 8$.

The second conditions are equivalents because Δ has real Fourier coefficients. Our function is of the form

$$\Phi = \alpha E_2^2 E_4^2 + \beta E_2 E_4 E_6 + \gamma E_4^3 + \delta E_6^2,$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. It is immediate that the Fourier coefficients of Φ are real if $\alpha, \beta, \gamma, \delta$ are so. This is the case here because all our equations will be real and linear.

All the Eisenstein series have non-zero Fourier coefficients for $m \geq 0$. Hence, we have two equations on $\alpha, \beta, \gamma, \delta$ saying that f_0 and f_1 are zero. The first equation is easy since all Eisenstein series are normalized with a first Fourier coefficient equal to 1. Hence, it is just $\alpha + \beta + \gamma + \delta = 0$. The second equation can be written by hand but it is easier to use a computer algebra system. We speak more about that kind of software bellow.

The two other equations are given using the transformation formula of the Eisenstein series, and in particular E_2. Recall that E_4 and E_6 are modular forms on $\Gamma(1)$ and that

$$E_2(-1/t)t^{-2} = E_2(t) - \frac{6i}{\pi t}.$$

Hence, the transformation formula for Φ is

$$\begin{aligned}\Phi(i/t)t^{-10} &= \alpha t^2 \left(E_2(t) - \frac{6}{\pi t}\right)^2 E_4(t)^2 + \beta t^2 \left(E_2(t) - \frac{6}{\pi t}\right) E_4(t)E_6(t) + \gamma t^2 E_4(t)^3 + \delta t^2 E_6(t)^2 \\ &= \Phi(it)t^2 - \frac{6t}{\pi}\left(2\alpha E_2(t)E_4(t)^2 + \beta E_4(t)E_6(t)\right) + \frac{36\alpha}{\pi^2}E_4(t)^2.\end{aligned} \tag{4.5}$$

To get the right expansion, we have to ensure two things. First, we want a constant factor $c \neq 0$. This is only the case if $\alpha \neq 0$. Since we are searching our function only up to a multiplicative constant, we can suppose that $\alpha = 1$. The second condition is for the factors in t. They are given by $\frac{6it}{\pi}\left(2\alpha E_2(t)E_4(t)^2 + \beta E_4(t)E_6(t)\right)$. We do not want a t factor but only a $te^{-2\pi t}$ one. This can be done by canceling the first term of the Fourier transforms. Since the Eisenstein series are normalized, this gives the equation $2\alpha + \beta = 0$. Looking at the Fourier expansion of E_4 and E_6, we see that the second Fourier coefficient does not vanish in this case.

Thus, the coefficients must solve the following system of equation (in matrix form):

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 432 & -288 & 720 & -1008 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

This system is solvable by hand. However, for the next sections, the computations are not so easy because the coefficients become quite big. For that reason, we do our computations on a computer algebra system. Here, we use PARI/GP which is specialized in number theory, but there also exists other software like the famous Mathematica. All the codes are given in appendix B. The solution of this equation is $(\alpha, \beta, \gamma, \delta) = (1, -2, 0, 1)$. Hence, our $+1$ Fourier eigenfunction in 8 dimensions is

$$\phi(t) = \frac{E_2^2(t)E_4^2(t) - 2E_2(t)E_4(t)E_6(t) + E_6^2(t)}{\Delta}.$$

We also have that the transformation identity of ϕ is

$$\phi(-1/t)t^2 = t^2\phi(t) - \frac{6it}{\pi}\phi_1(t) - \frac{36}{\pi^2}\phi_2(t),$$

with

$$\phi_1(t) := \frac{2E_2(t)E_4(t)^2 - 2E_4(t)E_6(t)}{\Delta}, \qquad \phi_2(t) := \frac{E_4(t)^2}{\Delta}.$$

Using the computer algebra system, we can compute the first terms of the Fourier series of all these forms quickly. This will be useful later.

$$\begin{aligned} \phi(t) &= 518400q + 31104000q^2 + O(q^3), \\ \phi_1(t) &= 1440 + 406080q + 18835200q^2 + O(q^3), \\ \phi_2(t) &= q^{-1} + 504 + 73764q + 2695040q^2 + O(q^3). \end{aligned} \tag{4.6}$$

4.2 The -1 Fourier eigenfunction

Now, we deal with the construction of the -1 Fourier eigenfunction. We are looking for a function such that its Fourier transform is minus itself and which is a radial Schwartz function. For the zeros, we only ask that it has a simple zero at $\sqrt{2}$ and a double zero at 0. We explain why below. We set the eigenfunction in the form

$$b(x) = \sin(\pi|x|^2/2)^2 \int_0^{i\infty} \psi(t)e^{i\pi|x|^2 t}dt, \tag{4.7}$$

where $\psi \in \mathcal{M}_k^!(\Gamma(2))$. We don't need ψ to be a quasimodular form as in the last section. Instead, it is a weakly holomorphic modular form of weight k on the group $\Gamma(2)$. Having forms over $\Gamma(2)$ allows us to have modular forms ϕ such that $\phi(t+1) = -\phi(t)$. This minus sign is useful for the Fourier transform. But since ψ is not a quasimodular form as ϕ in last section, we need to add a *had oc* condition for the Fourier transform. Note that for $\Gamma(2)$, the Fourier series of the modular forms are in $q^{1/2} = e^{i\pi t}$.

4.2.1 The zeros and the imaginary values

In the next section, we will take a linear combination of $a(x)$ and $b(x)$ and we don't want the resultant function to vanish at 0. Hence, it is not a problem (and actually better) if ψ vanishes at 0. The non-vanishing of $a(x)$ at 0 ensures that the final function does not vanish at this point. However, it is useful to have a similar expansion for ϕ and ψ. We will use it to cancel the biggest term in 4.2 when we combines $a(x)$ and $b(x)$.

In section 4.1.1, we used the quasimodularity of ϕ to cancel the zero at 0 with a pole of order 4. Looking at the computations of this section, we see that if

$$\psi(it) = ce^{2\pi t} + d + O(e^{-\pi t})$$

as $t \to \infty$, with $c, d \neq 0$, we have the right cancellation for the zeros at 0 and $\sqrt{2}$. With this expansion, the integral in equation 4.7 converges for $|x| > \sqrt{2}$. Again, we can rewrite $b(x)$ as

$$b(x) = i\sin(\pi|x|^2/2)^2 \left(\frac{c}{\pi(|x|^2 - 2)} + \frac{d}{\pi|x|^2} + \int_0^\infty \left(\psi(it) - (ce^{2\pi t} + d)\right) e^{-\pi|x|^2 t} dt \right). \tag{4.8}$$

Here, the integral converges for all x and $b(r)$ is analytic as a function of $r = |x|$. As for the +1 Fourier eigenfunction, if all the Fourier coefficients of ψ are real, then $b(x)$ takes purely imaginary values.

4.2.2 The Fourier transform and the Schwartz function

In the same way as in section 4.1.2, we expand the squared sine and deform the integration paths to make them pass all through i. This gives the form:

$$\begin{aligned} b(x) &= 2\int_0^{i\infty} \psi(t)e^{i\pi|x|^2 t}dt - \int_1^{i\infty+1} \psi(t-1)e^{i\pi|x|^2 t}dt - \int_{-1}^{i\infty-1} \psi(t+1)e^{i\pi|x|^2 t}dt \\ &= 2\int_0^{i} \psi(t)e^{i\pi|x|^2 t}dt - \int_1^{i} \psi(t-1)e^{i\pi|x|^2 t}dt - \int_{-1}^{i} \psi(t+1)e^{i\pi|x|^2 t}dt \\ &+ \int_i^{i\infty} (2\psi(t) - \psi(t-1) - \psi(t+1))e^{i\pi|x|^2 t}dt \end{aligned}$$

Note that $\Gamma(2)$ contains the translation $t \mapsto t-2$. Hence, we have $\psi(t-1) = \psi((t+1)-2) = \psi(t+1)$. So the last integral is

$$\int_i^{i\infty} 2(\psi(t) - \psi(t+1))e^{i\pi|x|^2 t}dt.$$

We denote $\tilde{\psi}(t) = 2\psi(t) - 2\psi(t+1)$. The Fourier transform of $b(x)$ is

$$\begin{aligned} \hat{b}(x) = 2\int_0^{i} \psi(t)t^{-n/2}e^{-i\pi|x|^2/t}dt - \int_1^{i} \psi(t+1)t^{-n/2}e^{-i\pi|x|^2/t}dt - \int_{-1}^{i} \psi(t+1)t^{-n/2}e^{-i\pi|x|^2/t}dt \\ + \int_i^{i\infty} \tilde{\psi}(t)t^{-n/2}e^{-i\pi|x|^2/t}dt. \end{aligned}$$

Making the change of variable $t \mapsto -1/t$, we get

$$\begin{aligned} \hat{b}(x) = -2\int_i^{i\infty} \psi(-1/t)t^{-k}e^{i\pi|x|^2}dt - \int_{-1}^{i} \psi(-1/t+1)t^{-k}e^{i\pi|x|^2 t}dt - \int_1^{i} \psi(-1/t+1)t^{-k}e^{i\pi|x|^2/t}dt \\ - \int_0^{i} \tilde{\psi}(-1/t)t^{-k}e^{i\pi|x|^2 t}dt. \end{aligned}$$

Identifying the integral of b and $\hat{b}$ with respect to their bounds, we find the following conditions on ψ: let $T : t \mapsto t+1$ and $S : t \mapsto -1/t$ two transformations of $\mathrm{SL}(2,\mathbb{Z})\backslash\Gamma(2)$. Then $\hat{b} = -b$ if $\psi[T]_k[S]_k = -\psi[T]_k$ and $\tilde{\psi} = 2\psi[S]_k$. The condition on $\tilde{\psi}$ can be rewritten as

$$\psi(-1/t) + \psi(t+1) = \psi(t).$$

or equivalently as $\psi[S]_k + \psi[T]_k = \psi$. Applying $[S]_k$ to this equation also gives that $\psi[T]_k[S]_k = -\psi[T]_k$.

This gives another form for $b(x)$:

$$b(x) = 2\int_0^i \psi(t)e^{i\pi|x|^2 t}dt - \int_1^i \psi(t-1)e^{i\pi|x|^2 t}dt - \int_{-1}^i \psi(t+1)e^{i\pi|x|^2 t}dt \\ + 2\int_i^{i\infty} \psi(-1/t)e^{i\pi|x|^2 t}dt \tag{4.9}$$

The same computations as before give that ψ is a Schwartz function. The crucial point was that the Fourier series of ϕ begins at a strictly positive index. Here, we did not invert ψ is the definition of $b(x)$. Hence, our condition is that the Fourier series of ψ is of the form

$$\psi(-1/t)t^{-k} = \sum_{m=1}^{\infty} f_{m/2}e^{i\pi mt}.$$

In this case, the integrals in equation 4.9 converge for all x. $b(x)$ is also an analytic function as a function of $r = |x|$. Hence, forms 4.8 and 4.9 are equal everywhere by analytic continuation.

Putting everything together, we get:

Theorem 4.4. *Let n a multiple of 8, $k = 2 - n/2$ and ψ a weakly holomorphic modular form of weight k on $\Gamma(2)$. If ψ satisfies:*

1. *ψ grows like $\psi(it) = ce^{2\pi t} + d + O(e^{-\pi t})$ as $t \to \infty$, with $c, d \neq 0$.*
2. *ψ has real Fourier coefficients.*
3. *$\psi(-1/t) + \psi(t+1) = \psi(t)$*
4. *$\psi(-1/t)t^{-k} = \sum_{m=1}^{\infty} f_{m/2}e^{i\pi mt}$*

Then $b(x)$ is a -1 Fourier eigenfunction taking values in $i\mathbb{R}$ and a radial Schwartz function with a simple zero at $|x| = \sqrt{2}$ and double zeros at 0 and $|x| = \sqrt{2k}$ for all $k \geq 2$.

4.2.3 Computation of the function

Now, we search for a function satisfying all the conditions of theorem 4.4. As in last section, we multiply ψ by Δ to get a holomorphic modular form of weight 10, denoted $\Psi := \Delta\psi$. From corollary 3.17, we know that a basis of $\mathcal{M}_{10}(\Gamma(2))$ is given by modular forms of the shape $\theta_{01}^{20-4j}\theta_{10}^{4j}$, $j = 0, \dots 5$. The conditions on Ψ are the following:

Proposition 4.5. *Let Ψ is a modular form of weight 10 and group $\Gamma(2)$ such that:*

1. *Ψ grows like $\Psi(it) = c + de^{-2\pi t} + O(e^{-3\pi t})$ as $t \to \infty$, with $c, d \neq 0$.*
2. *Ψ has real Fourier coefficients.*
3. *$\Psi(-1/t) + \Psi(t+1) = \Psi(t)$*
4. *$\Psi(-1/t)t^{-10} = \sum_{m=3}^{\infty} f_{m/2}e^{i\pi mt}$*

Then $\psi = \Psi/\Delta$ satisfies the conditions of theorem 4.4 for $n = 8$.

The space $\mathcal{M}_{10}(\Gamma(2))$ has 6 dimensions. As for the +1 Fourier eigenfunction, we create a system of linear equations to find Ψ. Once more, Ψ has real Fourier coefficients because all the equations are real, as well as the Fourier coefficients of θ_{01} and θ_{10}. The first three equations are given by canceling the Fourier coefficients of $\Psi(i/t)$. Another equation is determined by $c \neq 0$ in the first condition. As before, we are searching for a function only up to a constant. Hence, we set the normalization $c = 1$. The last equations are given by equating the first Fourier coefficients of $\Psi(-1/t)+\Psi(t+1)$ and $\Psi(t)$. Some of these equations are actually redundant, so we add equations up to the fourth coefficient.

We get a linear system of 8 equations and of rank 6. This system is a bit trickier to solve using PARI, because there is no implementation of θ_{01} and θ_{10} in it. The interested reader can refer to the appendix B for more information. Anyway, the solution of the system gives

$$\psi = \frac{5\theta_{01}^{12}\theta_{01}^{8} + 5\theta_{01}^{16}\theta_{01}^{4} + 2\theta_{01}^{20}}{2\Delta}.$$

The last condition we did not check before is that $d \neq 0$. This is visible in the expansion below.

$$\psi(t) = q^{-1} + 144 - 5120q^{1/2} + 70524q - 626688q^{3/2} + 4265600q^2 + O(q^{5/2}), \tag{4.10}$$
$$\psi(-1/t)t^2 = -10240q^{1/2} - 1253376q^{3/2} + O(q^{5/2}).$$

4.3 Proof of theorem 1.2

Finally, we can prove theorem 1.2. We recall it here:

Theorem 1.2 (Viazovska)**.** *The density of a sphere packing in 8 dimensions is at most $\frac{\pi^4}{384}$ and the unique periodic packing achieving it is the E_8 lattice packing.*

This theorem follows from the following proposition using the method of Cohn and Elkies:

Proposition 4.6. *There exists a function $f_8 : \mathbb{R}^8 \to \mathbb{R}$ such that:*

1. $f_8(0) = \hat{f}_8(0) = 1$.
2. $f_8(x) \geq 0$ *for all* $x \in \mathbb{R}^8$ *such that* $|x| \geq \sqrt{2}$.
3. $\hat{f}_8(x) \leq 0$ *for all* $x \in \mathbb{R}^8$.
4. f_8 *is radial and a Schwartz function.*
5. f_8 *has zeros at all non-zero vector lengths of* Λ_8. *These zeros are double zeros for all length except at* $\sqrt{2}$, *where it is a simple one.*

Therefore, using this function in theorem 2.8 and theorem 2.9 implies that theorem 1.2 holds.

We create f_8 as a linear combination of the function $a(x)$ and $b(x)$ constructed in the last sections. This immediately proves all the conditions on f_8 except the inequalities of conditions 2. and 3. The proof of these inequalities is not difficult but quite long, so we only sketch it. All the details are in Viazovska's paper [2].

Proof (Sketch). we create a function f_8 as a linear combination of $a(x)$ and $b(x)$:

$$f_8(x) = iAa(x) + iBb(x),$$

with constants $A, B \in \mathbb{R}$. Since our functions are taking values in $i\mathbb{R}$, we added a factor i. It is easy to determine A. $b(x)$ vanishes at 0 and $\hat{b}(x) = -b(x)$, so the values of $f_8(0)$ and $\hat{f}_8(0)$ depend only on $a(x)$. Since $a(x)$ is a +1 eigenfunction, we automatically have that $f_8(0) = \hat{f}_8(0)$. We normalize f_8 with $A = \frac{1}{ia(0)} = -4 \cdot \frac{\pi}{6} \cdot \frac{1}{1440} = -\frac{\pi}{2160}$. This value comes from equation 4.3 combined with the expansion 4.6.

We need to find the right value of B such that the two inequalities of theorem 2.8 holds. These inequalities are:

$$\begin{aligned} &iAa(x) + iBb(x) \le 0 && \forall x \in \mathbb{R}^8 \text{ s.t. } |x| > \sqrt{2}, \\ &iAa(x) - iBb(x) \ge 0 && \forall x \in \mathbb{R}^8. \end{aligned}$$

By definitions of $a(x)$ and $b(x)$, these inequalities are equivalent to:

$$\begin{aligned} &A\phi(i/t)t^2 - B\psi(it) \le 0 && \forall t \ge 0, \\ &A\phi(i/t)t^2 + B\psi(it) \ge 0 && \forall t \ge 0. \end{aligned} \tag{4.11}$$

Note that this equivalence is valid only for $|x| > \sqrt{2}$, because the integrals in equations 4.1 and 4.7 does not converge for $|x| \le \sqrt{2}$. This is not a problem for the first inequality. For the second one, we will see that we can extend the convergence of $iAa(x) - iBb(x)$ to all $x \neq 0$ by choosing the right value for B.

Looking at inequalities 4.11, we see that we need $B\psi(x) \ge 0$. By definition, θ_{01} and θ_{10} have real Fourier coefficients. Therefore, θ_{01}^4 and θ_{10}^4 take positive values on $i\mathbb{R}_{\ge 0}$. From its definition as a product, the modular discriminant Δ is also always positive on $i\mathbb{R}_{\ge 0}$. In view of this, we have

$$\psi(it) \ge 0 \quad \forall t \ge 0.$$

Hence, B must be positive. Moreover, if these inequalities hold for some value of B, then it also holds for B multiplied by any number bigger than 1. When $t \to \infty$, $\phi(i/t)t^2 = \frac{36}{\pi^2}e^{2\pi t} + O(t)$ and $\psi(it) = e^{2i\pi t} + O(1)$. Therefore, to have the second inequality, we need at least to cancel this term. This implies that $B \ge \frac{\pi}{2160}\frac{36}{\pi^2} = \frac{1}{60\pi}$. We fix $B = \frac{1}{60\pi}$.

For this value of B, the term in $e^{2\pi t}$ is canceled in $A\phi(i/t)t^2 + B\psi(it)$. Hence, the smallest terms in this equation are the constant term and the term in t. Looking back at section 4.1.1 and in particular equations 4.3 and 4.8, we see that the integral

$$\int_0^\infty (\phi(i/t)t^{-2} + \psi(it))e^{-\pi|x|^2 t} dt$$

converges for all $x \neq 0$. In particular the equivalence between the third inequality of the proposition and the second inequality of 4.11 is proven.

Looking at equations 4.6 and 4.10, we get the following expansion of ϕ and ψ around ∞ and 0:

$$
\begin{aligned}
&\text{As } t \to \infty: \\
&\phi(i/t)t^2 = \frac{36}{\pi^2}(e^{2\pi t} + 504 + 73764e^{-2\pi t} + 2695040e^{-4\pi t}) - \frac{6t}{\pi}(1440 + 406080e^{-2\pi t} + 18835200e^{-4\pi t}) \\
&\qquad + t^2(518400e^{-2\pi t} + 31104000e^{-4\pi t}) + O(t^2e^{-5\pi t}), \\
&\psi(it) = e^{2\pi t} + 144 - 5120e^{-\pi t} + 70524e^{2\pi t} - 626688e^{-3\pi t} + 4265600e^{-4\pi t} + O(e^{-5\pi t}). \\
&\text{As } t \to 0: \\
&\phi(i/t)t^2 = t^2(518400e^{-2\pi/t} + 3110400e^{-4\pi/t}) + O(t^2e^{-5\pi/t}), \\
&\psi(it) = t^2(10240e^{-\pi/t} + 1253376e^{-3\pi/t}) + O(t^2e^{-6\pi/t}).
\end{aligned}
$$

With the values $A = -\frac{\pi}{2160}$ and $B = \frac{1}{60\pi}$, we have the following behavior at 0 and ∞:

$$
\begin{aligned}
&\text{As } t \to \infty: \\
&A\phi(i/t)t^2 - B\psi(it) = -\frac{1}{30\pi}e^{2\pi t} + O(t), \\
&A\phi(i/t)t^2 + B\psi(it) = 4t + O(1). \\
&\text{As } t \to 0: \\
&A\phi(i/t)t^2 - B\psi(it) = -\frac{512}{3\pi}t^2e^{-\pi/t} + O(t^2e^{-\pi/t}), \\
&A\phi(i/t)t^2 + B\psi(it) = \frac{512}{3\pi}t^2e^{-\pi/t} + O(t^2e^{-\pi/t}).
\end{aligned}
$$

Hence, around 0 and ∞, the inequalities 4.11 are valid. The last part of the proof consists in checking these inequalities everywhere. We only sketch this part.

To prove the inequalities, we split the interval $(0, \infty)$ into $(0, 1]$ and $[1, \infty)$. For $t \in [1, \infty)$, we consider the functions $\delta(t) := A\phi(i/t)t^2 - B\psi(it)$ and $\tilde{\delta}(t) = A\phi(i/t)t^2 + B\psi(it)$. In the same way as above, we combine the Fourier series of ϕ and ψ to get the series of δ and $\tilde{\delta}$. We denote $\delta_n(t)$ the truncated expansion of δ up to an error of size $O(t^2e^{-\pi nt})$. We denote the error $R_n(t) = |\delta(t) - \delta_n(t)|$. Using the explicit formula for the Fourier coefficients of the Eisenstein series and the Thetanullwerte, or some effective version of theorem 3.5, we get effective constants for the bound on the Fourier coefficients of ϕ and ψ. The discussion at the beginning of appendix A in [3] gives details on the first idea. The bounds given by the second method are in [2]. Using these bounds, we get a bound on the size of the error. Finally, for some n big enough (in her paper, Viazovska took $n = 6$), we check two things using interval arithmetic. First, that the approximation satisfies the inequalities 4.11. Second, that the error term is always smaller than the approximation term, i.e. $R_n(t) \leq |\delta(t)|$ for all $t \in [1, \infty)$. This implies that the inequality for δ holds between 1 and ∞. This method gives the same result for $\hat{\delta}$.

For $t \in (0, 1]$, we consider the expansions of ϕ and ψ as $t \to 0$ and do the same reasoning. The approximation up to $n = 6$ also works. □

5 The sphere packing problem in dimension 24

This chapter follows the same ideas as the last one. However, most of the work is already done. We only need to make small adjustments to adapt our proof to this situation. One of them is that the Leech lattice that we consider in 24 dimensions has no vector of length $\sqrt{2}$. Another is that

$k = -10$, so we need to multiply our forms by Δ^2 to get a holomorphic form. Also, in the last section, a new problem arises in this dimension. It is about the convergence of the integrals.

5.1 The +1 Fourier eigenfunction

We take again a function $a(x)$ of the form

$$a(x) = \sin(\pi|x|^2/2)^2 \int_0^{i\infty} \phi\left(\frac{-1}{t}\right) t^{-k} e^{i\pi|x|^2 t} dt, \tag{5.1}$$

where ϕ is a weakly holomorphic quasimodular form of weight $k+2 = -8$ on $\mathrm{SL}(2,\mathbb{Z})$. We want that $a(x)$ satisfies the following conditions:

1. It has double zeros at all vector length $\sqrt{2k}$ for $k \geq 3$, a simple zero at $|x| = 2$ and does not vanish at 0.
2. It is imaginary valued.
3. Its 24-dimensional Fourier transform is itself.
4. It is a radial Schwartz function.

The only difference with last chapter is the location of the zeros. Looking at section 4.1.1, we see that it is enough that ϕ has an expansion of the form

$$\phi(i/t)t^{-10} = ce^{4\pi t} + (dt + f)e^{2\pi t} + (gt + h) + O(t^2 e^{-2\pi t}),$$

with $c, d, g \neq 0$. The term in c gives a simple pole that cancels once the double zero at $|x| = 2$ and the terms in d and g give poles of order 2 respectively 4 that cancel the zeros at $|x| = 0$ and $\sqrt{2}$. Hence, the integral in 5.1 converges only for $|x| > 2$ and we can rewrite $a(x)$ as

$$\begin{aligned} a(x) = -i\sin(\pi|x|^2/2)^2 &\left(\frac{c}{\pi(|x|^2-4)} + \frac{d}{\pi^2(|x|^2-2)^2} + \frac{f}{\pi(|x|^2-2)} + \frac{g}{\pi^2|x|^4} + \frac{h}{\pi|x|^2} \right. \\ &\left. + \int_0^{\infty} \left(\phi\left(\frac{i}{t}\right) t^{-k} - (ce^{4\pi t} + (dt+f)e^{2\pi t} + gt + h) \right) e^{-\pi|x|^2 t} dt \right). \end{aligned} \tag{5.2}$$

The integral above converges for all x. It also says that $a(x)$ has purely imaginary values if the Fourier series of ϕ has real coefficients. The computations of section 4.1.2 about the Fourier transform of $a(x)$ are still valid, but only for $|x| > 2$ instead of $|x| > \sqrt{2}$. It says that $a(x)$ is a +1 Fourier eigenfunction because ϕ is a quasimodular form of depth two. Finally, the reasoning of section 4.1.3 that prove that $a(x)$ is a Schwartz function is also valid. It also gives the condition that the Fourier series of ϕ begin at $m = 1$. We also have that the two forms of $a(x)$ are equal everywhere because they are analytic continuations of equation 5.1. Hence, we can go directly to the explicit computation of the function $a(x)$.

Since $k = -10$, the function $\Phi := \Delta^2\phi$ should be a quasimodular form of weight 16 and depth 2 on $\Gamma(1)$. Rewriting theorem 4.2 and proposition 4.31 in 24 dimensions, we get the following conditions on Φ:

Proposition 5.1. *Let Φ a quasimodular form of depth 2 and weight 16 on $\Gamma(1)$ such that:*

1. Φ grows like $\Phi(i/t)t^{-14} = c + (dt+f)e^{-2\pi t} + (gt+h)e^{-4\pi t} + O(t^2e^{-6\pi t})$ as $t \to \infty$, with $c, d, g \neq 0$.

2. Φ has real Fourier coefficients.

3. The Fourier series of Φ is $\Phi(t) = \sum_{m=3}^{\infty} f_m q^m$.

Then for $\phi := \Phi/\Delta^2$, $a(x)$ is a $+1$ Fourier eigenfunction taking values in $i\mathbb{R}$ and a radial Schwartz function with a simple zero at $|x| = 2$ and double zeros at $|x| = \sqrt{2k}$ for all $k \geq 3$.

Theorem 3.10 tells us that the space of quasimodular forms of weight 16 and depth 2 over $\Gamma(1)$ is of dimension 5. It is spanned by $E_2^2E_4^3$, $E_2^2E_6^2$, $E_2E_4^2E_6$, E_4^4 and $E_4E_6^2$. The Fourier series of Φ gives 3 linear equations. We add the normalization $c = 24$. This is the right value to get an integer solution. A fifth equation is that there is no term in t in the equation for $\Phi(i/t)t^{-14}$. The same calculation as equation 4.5 gives the equation $2\alpha E_2E_4^3 + 2\beta E_2E_6^2 + \gamma E_4^2E_6 = 0$ for this condition, where α, β, γ are the coefficient of $E_2^2E_4^3$, $E_2^2E_6^2$ and $E_2E_4^2E_6$. This gives the solution

$$\phi = \frac{49E_2^2E_4^3 - 25E_2^2E_6^2 - 48E_2E_4^2E_6 - 25E_4^4 + 49E_4E_6^2}{\Delta^2}.$$

Using the expansions given below, one can check that Φ follows the expansion of the first condition. The transformation formula 3.2 of E_2 implies that

$$\phi(-1/t)t^{10} = t^2\phi(t) - \frac{6it}{\pi}\phi_1(t) - \frac{36}{\pi^2}\phi_2(t),$$

with

$$\phi_1 = \frac{98E_2E_4^3 - 50E_2E_6^2 - 48E_4^2E_6}{\Delta^2}, \qquad \phi_2 = \frac{49E_4^3 - 25E_6^2}{\Delta^2}.$$

These functions have the following expansion:

$$\begin{aligned} \phi &= 3657830400q + 314573414400q^2 + O(q^3), \qquad (5.3)\\ \phi_1 &= 120960q^{-1} + 18869760 - 3281886720q + 214680775680q^2 + O(q^3),\\ \phi_2 &= 24q^{-2} + 61632q^{-1} + 6198336 + 649114368q + 34850569824q^2 + O(q^3). \end{aligned}$$

5.2 The -1 Fourier eigenfunction

As in 8 dimensions, we set

$$b(x) = \sin(\pi|x|^2/2)^2 \int_0^{i\infty} \psi(t)e^{i\pi|x|^2t}dt, \qquad (5.4)$$

with ψ a weakly holomorphic modular form of weight $k = -10$ on $\Gamma(2)$. As above, the only modification we have to do is on the grow of ψ. If

$$\psi(it) = ce^{4\pi t} + de^{2\pi t} + f + O(e^{-\pi t}),$$

with $c, d, f \neq 0$, then ψ has a simple zero at $|x| = 0$, $\sqrt{2}$ and 2 and double zeros for $|x| = \sqrt{2k}$, $k \geq 3$. The other reasoning of section 4.2 are still valid. Therefore, we can reformulate theorem 4.4 and proposition 4.5 in dimension 24 in the following way:

Proposition 5.2. *Let* Ψ *a modular form of weight* 14 *on* $\Gamma(2)$ *such that:*

1. Ψ *grows like* $\Psi(it) = c + de^{-2\pi t} + fe^{-4\pi t} + O(e^{-5\pi t})$ *as* $t \to \infty$*, with* $c, d, f \neq 0$*.*
2. Ψ *has real Fourier coefficients.*
3. $\Psi(-1/t)t^{-14} + \Psi(t+1) = \Psi(t)$
4. $\Psi(-1/t)t^{-14} = \sum_{m=5}^{\infty} f_{m/2} e^{i\pi m t}$.

Then for $\psi := \Psi/\Delta^2$*,* $b(x)$ *is a* -1 *Fourier eigenfunction taking values in* $i\mathbb{R}$ *and a radial Schwartz function with a simple zero at* $|x| = 2$ *and double zeros at* $|x| = \sqrt{2k}$ *for all* $k \geq 3$*.*

The space of modular forms of weight 14 on $\Gamma(2)$ is of dimension 8 and its structure is given by theorem 3.17. Five equations comes from the Fourier series of $\Psi(-1/t)$. A condition is given by the normalization $c = 2$. Two other equations comes from the condition $\Psi(-1/t) + \Psi(t+1) = \Psi(t)$. This gives the solution

$$\psi = \frac{7\theta_{01}^{20}\theta_{10}^{8} + 7\theta_{01}^{24}\theta_{10}^{4} + 2\theta_{01}^{28}}{\Delta^2}.$$

Finally, the conditions $d, f \neq 0$ are visible in the expansion below:

$$\psi(t) = 2q^{-2} - 464q^{-1} + 172128 - 3670016q^{1/2} + 47238464q - 459276288q^{3/2} + O(q^2), \tag{5.5}$$

$$\psi(-1/t)t^{10} = -7340032q^{1/2} - 918552576q^{3/2} + O(q^{5/2}).$$

5.3 Proof of theorem 1.3

We now turn to the proof of theorem 1.3. We recall it here:

Theorem 1.3 (Cohn, Kumar, Miller, Radchenko, Viazovska)**.** *The density of a sphere packing in 24 dimensions is at most* $\frac{\pi^{12}}{12!}$ *and the unique periodic packing achieving it is the Leech lattice packing.*

In the same way as in the last chapter, this theorem follows by applying the method of Cohn and Elkies using the following proposition:

Proposition 5.3. *There exists a function* $f_{24} : \mathbb{R}^{24} \to \mathbb{R}$ *such that:*

1. $f_{24}(0) = \hat{f}_{24}(0) = 1$.
2. $f_{24}(x) \geq 0$ *for all* $x \in \mathbb{R}^{24}$ *such that* $|x| \geq 2$.
3. $\hat{f}_{24}(x) \leq 0$ *for all* $x \in \mathbb{R}^{24}$.
4. f_{24} *is radial and a Schwartz function.*
5. f_{24} *has zeros at all non-zero vector lengths of the Leech lattice. These zeros are double zeros for all length except at* 2*, where it is a simple one.*

Therefore, using this function in theorem 2.8 and theorem 2.9 implies that theorem 1.3 holds.

The proof follows the same strategy as the proof of proposition 4.6 at the end of last chapter. There is just one more difficulty at the end.

Proof (sketch). We construct f_{24} as a linear combination of $a(x)$ and $b(x)$:

$$f_{24} = iAa(x) + iBb(x).$$

We set $A = \frac{1}{ia(0)} = -4\frac{\pi}{6}\frac{1}{18869760} = -\frac{\pi}{28304640}$ and $B = -A\frac{36}{\pi^2}\frac{24}{2} = \frac{1}{65520\pi}$. Looking at the expansions 5.3 and 5.5, we see that this value of B cancels exactly the q^{-2} term in $A\phi(-1/t)t^{10} - B\psi(t)$.

We want to prove the two inequalities:

$$\begin{aligned} &iAa(x) + iBb(x) \leq 0 && \forall x \in \mathbb{R}^{24} \text{ s.t. } |x| > 2, \\ &iAa(x) - iBb(x) \geq 0 && \forall x \in \mathbb{R}^{24}. \end{aligned}$$

As in the proof of proposition 4.6, we reduce ourselves to the following inequalities:

$$\begin{aligned} &A\phi(i/t)t^{10} - B\psi(it) \leq 0 && \forall t \geq 0, \\ &A\phi(i/t)t^{10} + B\psi(it) \geq 0 && \forall t \geq 0. \end{aligned} \tag{5.6}$$

Unfortunately, in this case the inequalities 5.6 are not equivalent to the inequalities of the proposition. This is because $A\phi(i/t)t^{10} + B\psi(it)$ has a non-zero term in q^{-1}. Hence, the integral in $iAa(x) - iBb(x)$ converges only for $|x| > \sqrt{2}$. We come back to this issue at the end of the proof. Looking at the expansion 5.3 and 5.5, we see that:

$$\begin{aligned} &\text{As } t \to \infty: \\ &A\phi(i/t)t^{10} - B\psi(it) = -\frac{1}{16380\pi}e^{4\pi t} + O(te^{2\pi t}), \\ &A\phi(i/t)t^{10} + B\psi(it) = \frac{1}{39}te^{2\pi t} - \frac{10}{117}e^{2\pi t} + O(t). \\ &\text{As } t \to 0: \\ &A\phi(i/t)t^{10} - B\psi(it) = -\frac{65536}{585\pi}t^{10}e^{-\pi/t} + O(t^{10}e^{-2\pi/t}), \\ &A\phi(i/t)t^{10} + B\psi(it) = \frac{65536}{585\pi}t^{10}e^{-\pi/t} + O(t^{10}e^{-2\pi/t}). \end{aligned}$$

Hence, the inequalities 5.6 are valid around 0 and ∞. One needs to check the inequalities everywhere. We separate again the case $t \in (0, 1]$ and $t \in [1, \infty)$. For t in the second interval, we define $\delta = A\phi(i/t)t^2 - B\psi(it)$ and $\delta_n(t)$ which is the truncated series of δ up to an error of size $O(t^2e^{-\pi nt})$. We use a different method from the one of proposition 4.6. It is based on Sturm's theorem (theorem 2.50 in [17]) that gives the number of zeros of a polynomial in an interval. We approximate the error term $R_n(t) = |\delta(t) - \delta_n(t)|$ with $n = 50$. This gives a bound $R_n(t) \leq 10^{-50}|q|^6$ in all cases. We apply Sturm's theorem to δ_n, seen as a polynomial in $q^{1/2}$, plus or minus $10^{-50}q^6$. This gives that δ cannot change sign on the interval. Since the value of δ at ∞ is consistent with the inequalities 5.6, the inequality for δ hold for $t \in [1, \infty)$. We do the same computation for $\hat{\delta}$ and for $t \in (0, 1]$. The approximation up to $n = 50$ always works and gives a error term smaller than $10^{-50}|q|^6$.

Applying Sturm's theorem to a polynomial of degree 49 is not feasible by hand. In [3], they used a computer algebra system for the calculations. In particular, they approximated every number appearing in the computation by rational ones. It allows them to do exact calculations. For example, π is approximated from above and below by rounding its 10th decimal. All the details of their proof is given in [3] and the attached PARI/GP program.

We have to deal with one last thing. Proving inequalities 5.6 does not provide a proof that $iAa(x) - iBb(x) \geq 0$ for $|x| \leq \sqrt{2}$. To do this, we split the integral in $a(x)$ and $b(x)$ into to pieces, one with $t \in (0,1]$ and the other with $t \in [1,\infty)$. The only problem is the term in $te^{2\pi t}$ at ∞. Hence, the computations above tell us that the integral on $(0,1]$ converges and satisfies the inequalities. For the second one, we have the following expansion:

$$A\phi(i/t)t^{10} + B\psi(it) = \frac{1}{39}te^{2\pi t} - \frac{10}{117}e^{2\pi t} + O(t),$$

as $t \to \infty$. The remaining term $O(t)$ does not cause troubles. In section 4.1.1, we extended the convergence of the integral by computing the integral for $|x|$ big enough. Here, we do the same with the problematic terms:

$$\int_1^\infty \left(\frac{1}{39}te^{2\pi t} - \frac{10}{117}e^{2\pi t}\right) e^{-\pi|x|^2 t} dt = \frac{(10-3\pi)(2-|x|^2)+3}{117\pi^2(2-|x|^2)^2} e^{-\pi(|x|^2-2)} =: S(x).$$

Then we prove that the function

$$S(x) + \int_1^\infty \left((A\phi(i/t)t^{10} + B\psi(it)) - \frac{1}{39}te^{2\pi t} + \frac{10}{117\pi}e^{2\pi t}\right) e^{-\pi|x|^2 t} dt$$

is negative for $0 < |x| < \sqrt{2}$ in the same way as before. This is the case for $S(x)$. For the integral, it is equivalent to prove that

$$A\phi(i/t)t^{10} + B\psi(it) \geq \frac{1}{39}te^{2\pi t} - \frac{10}{117\pi}e^{2\pi t}.$$

This is done in the same way as before by approximating the Fourier series up to q^{50} and using Sturm's theorem. □

A Matrices of the lattices

We give here a basis for the E_8 and the Leech lattices as the rows of a matrix. Any other basis of the corresponding lattice can be deduced by multiplication on the right by an element of $GL_n(\mathbb{Z})$ for $n = 8$, respectively $n = 24$. Basis for Λ_8 and Λ_{24} are given by the matrix M_8, respectively M_{24}.

$$M_8 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad M_{24} = \frac{1}{\sqrt{8}} \begin{pmatrix} 8&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 2&2&2&2&2&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 2&2&2&2&0&0&0&0&2&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0 \\ 2&2&0&0&2&2&0&0&2&2&0&0&2&2&0&0&0&0&0&0&0&0&0&0 \\ 2&0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&0&0&0&0&0&0&0&0 \\ 2&0&0&2&2&0&0&2&2&0&0&2&2&0&0&2&0&0&0&0&0&0&0&0 \\ 4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0 \\ 2&0&2&0&2&0&0&2&2&2&0&0&0&0&0&0&2&2&0&0&0&0&0&0 \\ 2&0&0&2&2&2&0&0&2&0&2&0&0&0&0&0&2&0&2&0&0&0&0&0 \\ 2&2&0&0&2&0&2&0&2&0&0&2&0&0&0&0&2&0&0&2&0&0&0&0 \\ 0&2&2&2&2&0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&2&0&0&0 \\ 0&0&0&0&0&0&0&0&2&2&0&0&2&2&0&0&2&2&0&0&2&2&0&0 \\ 0&0&0&0&0&0&0&0&2&0&2&0&2&0&2&0&2&0&2&0&2&0&2&0 \\ -3&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{pmatrix}.$$

One easily computes (using a computer) that these matrices both have a determinant equal to one, and that the products between them and their transpose produce integral matrices.

B Computer codes

PARI/GP is a computer algebra system written in C and available for free under public license at the address `pari.math.u-bordeaux.fr`. It is specialized in number theory. One notable difference with usual programming language is that arrays are considered as matrices and so their index begins at 1 and not 0. Since all programs follow the same idea, only the first one is commented in detail.

B.1 +1 Fourier eigenfunction in 8 dimensions

For the positive Fourier eigenfunction, we use the PARI implementation of modular forms.

Dimension8Positive.gp

```
\\This program find the +1 Fourier eigenfunction in 8 dimensions.

\\Initialize a space of modular forms, required for computations
ModularSpace=mfinit([1,0]);

\\The size of the Fourier expansions we consider
n=100;
\\The size of the Fourier expansions we display
nDisplay=5;

\\We begin by setting a basis of the quasimodular forms of weight 12 and depth 2.
   We take a Fourier expansion of n terms of this basis
\\These variables define the Eisenstein series of weight 2,4 and 6 and the modular
   discriminant
Eisenstein2=mfEk(2);
Eisenstein4=mfEk(4);
Eisenstein6=mfEk(6);
Discriminant=mfDelta();

\\The basis of quasimodular forms of weight 12 and depth 2
Basis=[{
  mfmul(mfpow(Eisenstein2,2),mfpow(Eisenstein4,2)),
  mfmul(mfmul(Eisenstein2,Eisenstein4),Eisenstein6),
  mfpow(Eisenstein4,3),
  mfpow(Eisenstein6,2)
}];

\\Fourier series of the basis. SeriesBasis[i][j] is the j-th Fourier factor of the
   i-th function of the basis
SeriesBasis=vector(4);
for(i=1,4,{
  SeriesBasis[i]=mfcoefs(Basis[i],n);
});

\\We set our equation system using the factors of the Fourier expansion of the
   basis. We denote fj for the j-th Fourier factor of Delta*phi(t).
\\f0=0
Condition1=[SeriesBasis[1][1],SeriesBasis[2][1],SeriesBasis[3][1],SeriesBasis
   [4][1]];
\\f1=0
Condition2=[SeriesBasis[1][2],SeriesBasis[2][2],SeriesBasis[3][2],SeriesBasis
   [4][2]];
\\Normalization c=1
```

```
Condition4=[1,0,0,0];
\\Condition for the t factor
Condition3=[2,1,0,0];

\\The matrix of the equation system
M=matconcat([Condition1;Condition2;Condition3;Condition4]);

\\The vector of constant terms. The last one correspond to the normalization
    condition
A=[0,0,0,1]~;

\\Solve M*X=A for X=(alpha,beta,gamma,delta)
X=matsolve(M,A);

\\Display the solution in the console
printf("\nThe solution of the system is:");
printf(concat("X=(alpha,beta,gamma,delta)=",X));

\\Now, we compute the modular forms Delta*phi and phi
DeltaPhi=mflinear([Basis[1],Basis[2],Basis[3],Basis[4]],X);
Phi=mfdiv(DeltaPhi,Discriminant);

\\We do the same for phi1 and phi2. The command mfshift(F,s) divide the modular
    form F by q^s
Phi1=mfdiv(mflinear([Basis[1],Basis[2]],[2*X[1],X[2]]),mfmul(Eisenstein2,
    Discriminant));
Phi2=mfdiv(mfshift(mflinear([Basis[1]],[X[1]]),-1),mfmul(mfpow(Eisenstein2,2),
    Discriminant));

\\We compute the Fourier series of these forms and display them
SeriesPhi=mfcoefs(Phi,nDisplay);
SeriesPhi1=mfcoefs(Phi1,nDisplay);
SeriesPhi2=mfcoefs(Phi2,nDisplay);

printf("\nThe Fourier series of phi, phi1 and phi2 are:");
printf(concat("\nphi(t)=",Ser(SeriesPhi,q)));
printf(concat("\nphi1(t)=",Ser(SeriesPhi1,q)));
printf(concat(["\nphi2(t)=1/q[",Ser(SeriesPhi2,q),"]"]));
```

B.2 -1 Fourier eigenfunction in 8 dimensions

For the negative Fourier eigenfunction, we do not have a implementation of the theta functions θ_{01} and θ_{10} in PARI. Instead, we directly compute their Fourier series up to a big enough term. For the last condition

$$\psi(-1/t) + \psi(t+1) = \psi(t),$$

the equation system only checks it for the first Fourier coefficients. Using the transformation formulas for the Thetanullwerte, one can easily compute by hand that this equation is valid. Since we are working in a space of finite dimension, it is also possible to prove it by computing enough Fourier terms. The program does this and gives the error on its computations.

Dimension8Negative.gp

```
\\This program find the -1 Fourier eigenfunction in 8 dimensions.

n=100;
nDisplay=5;

\\We begin by defining the thetanullwerte functions. Since they are not implemented
    in PARI/GP, we approximate them up to the n-th Fourier term by a polynomial in
   r=q^1/2
\\These variables define the Eisenstein series, the modular discriminant and the 4
   th power of thetanullwerte functions. bernfrac(n) is the n-th Bernoulli number
q = r^2;
Eisenstein(k) = 1-2*k/bernfrac(k)*sum(m=1,n,sigma(m,k-1)*q^m)+O(q^n);
Discriminant = (Eisenstein(4)^3-Eisenstein(6)^2)/1728;
Theta004=(sum(m=-n,n,r^(m^2)) + O(r^(2*n)))^4;
Theta014=(sum(m=-n,n,(-1)^m*r^(m^2)) + O(r^(2*n)))^4;
Theta104=r*(sum(m=-n,n,r^(m^2+m))+O(r^(2*n)))^4;

\\Basis of modular form of weight 10 on Gamma(2). T and S designate the usual
   transformations t->t+1 and t->-1/t
Basis=vector(6);
BasisT=vector(6);
BasisS=vector(6);
for(i=1,6,{
  Basis[i]=Theta014^(i-1)*Theta104^(6-i);
  BasisT[i]=Theta004^(i-1)*(-Theta104)^(6-i);
  BasisS[i]=(-Theta104)^(i-1)*(-Theta014)^(6-i);
});

\\Equation system. We denote fSj for the j-th Fourier factor of Delta*psi(-1/t).
Condition=vector(8);
for(i=1,8,{
  Condition[i]=vector(6);
});
for(j=1,6,{
  \\fS0=0
  Condition[1][j]=polcoef(BasisS[j],0);
  \\fS0.5=0
  Condition[2][j]=polcoef(BasisS[j],1);
  \\fS1=0
  Condition[3][j]=polcoef(BasisS[j],2);
  \\c=1
  Condition[4][j]=polcoef(Basis[j],0);
});

\\Condition psi(-1/t)+psi(t+1)=psi(t) for the four first coefficients
for(i=5,8,{
  for(j=1,6,
    Condition[i][j]=polcoef(BasisS[j],i)+polcoef(BasisT[j],i)-polcoef(Basis[j],i);
  );
});

M=matconcat([Condition[1];Condition[2];Condition[3];Condition[4];Condition[5];
    Condition[6];Condition[7];Condition[8]]);
A=[0,0,0,1,0,0,0,0]~;
```

```
X=matsolve(M,A);

printf("\nThe solution of the system is:");
printf(concat("X=",X));

\\Computation of the series of the modular forms
DeltaPsi=sum(i=1,6,X[i]*Basis[i]);
DeltaPsiS=sum(i=1,6,X[i]*BasisS[i]);
DeltaPsiT=sum(i=1,6,X[i]*BasisT[i]);

Psi=DeltaPsi/Discriminant;
PsiS=DeltaPsiS/Discriminant;
PsiT=DeltaPsiT/Discriminant;

printf("\nThe Fourier series of psi(t) and psi(-1/t) are (r=q^1/2):");
printf(concat("\npsi(t)=",Psi+O(q^nDisplay)));
printf(concat("\npsi(-1/t)=",PsiS+O(q^nDisplay)));

printf("\nThe error on the equation psi(-1/t)+psi(t+1)=psi(t) is at most:");
printf(concat("psi(-1/t)+psi(t+1)-psi(t)=",PsiS+PsiT-Psi));
```

B.3 +1 Fourier eigenfunction in 24 dimensions

Dimension24Positive.gp

```
\\This program find the +1 Fourier eigenfunction in 24 dimensions.

mf=mfinit([1,0]);
n=100;
nDisplay=5;

\\Eisenstein series and modular discriminant
Eisenstein2=mfEk(2);
Eisenstein4=mfEk(4);
Eisenstein6=mfEk(6);
Discriminant=mfDelta();

\\Basis of quasimodular forms of weight 16 and depth 2
Basis={[
  mfmul(mfpow(Eisenstein2,2),mfpow(Eisenstein4,3)),
  mfmul(mfpow(Eisenstein2,2),mfpow(Eisenstein6,2)),
  mfmul(mfmul(Eisenstein2,mfpow(Eisenstein4,2)),Eisenstein6),
  mfpow(Eisenstein4,4),
  mfmul(Eisenstein4,mfpow(Eisenstein6,2))
]};

\\Fourier series of the basis
SeriesBasis=vector(5);
for(i=1,5,{
  SeriesBasis[i]=mfcoefs(Basis[i],n)
});

\\Equation system
\\f0=0
```

```
Condition1=[SeriesBasis[1][1],SeriesBasis[2][1],SeriesBasis[3][1],SeriesBasis
    [4][1],SeriesBasis[5][1]];
\\f1=0
Condition2=[SeriesBasis[1][2],SeriesBasis[2][2],SeriesBasis[3][2],SeriesBasis
    [4][2],SeriesBasis[5][2]];
\\f2=0
Condition3=[SeriesBasis[1][3],SeriesBasis[2][3],SeriesBasis[3][3],SeriesBasis
    [4][3],SeriesBasis[5][3]];
\\c=24, if factor E2^2 -> 1, otherwise 0
Condition4=[1,1,0,0,0];
\\Cond for t, if E2^2 -> 2, if E2^1 -> 1
Condition5=[2,2,1,0,0];

M=matconcat([Condition1;Condition2;Condition3;Condition4;Condition5]);

A=[0,0,0,24,0]~;
X=matsolve(M,A);

printf("\nThe solution of the system is:");
printf(concat("X=",X));

\\Computation of the modular forms and their series
DeltaPhi=mflinear([Basis[1],Basis[2],Basis[3],Basis[4],Basis[5]],X);
Phi=mfdiv(DeltaPhi,mfpow(Discriminant,2));
Phi1=mfdiv(mfshift(mflinear([Basis[1],Basis[2],Basis[3]],[2*X[1],2*X[2],X[3]]),-1),
    mfmul(Eisenstein2,mfpow(Discriminant,2)));
Phi2=mfdiv(mfshift(mflinear([Basis[1],Basis[2]],[X[1],X[2]]),-2),mfmul(mfpow(
    Eisenstein2,2),mfpow(Discriminant,2)));

SeriesPhi=mfcoefs(Phi,nDisplay);
SeriesPhi1=mfcoefs(Phi1,nDisplay);
SeriesPhi2=mfcoefs(Phi2,nDisplay);

printf("\nThe Fourier series of phi, phi1 and phi2 are:");
printf(concat("\nphi(t)=",Ser(SeriesPhi,q)));
printf(concat(["\nphi1(t)=1/q[",Ser(SeriesPhi1,q),"]"]));
printf(concat(["\nphi2(t)=1/q^2[",Ser(SeriesPhi2,q),"]"]));
```

B.4 -1 Fourier eigenfunction in 24 dimensions

Dimension24Negative.gp

```
\\This program find the -1 Fourier eigenfunction in 24 dimensions.

n=100;
nDisplay=5;

\\Eisenstein series, modular discriminant and 4th power of thetanullwerte functions
q = r^2;
Eisenstein(k) = 1-2*k/bernfrac(k)*sum(m=1,1000,sigma(m,k-1)*q^m)+O(q^n);
Discriminant = (Eisenstein(4)^3-Eisenstein(6)^2)/1728;
Theta004=(sum(m=-n,n,r^(m^2)) + O(r^(2*n)))^4;
Theta014=(sum(m=-n,n,(-1)^m*r^(m^2)) + O(r^(2*n)))^4;
Theta104=r*(sum(m=-n,n,r^(m^2+m))+O(r^(2*n)))^4;
```

```
\\Basis of modular form of weight 14 on Gamma(2).
Basis=vector(8);
BasisS=vector(8);
BasisT=vector(8);
for(i=1,8,{
  Basis[i]=Theta014^(i-1)*Theta104^(8-i);
  BasisS[i]=(-Theta104)^(i-1)*(-Theta014)^(8-i);
  BasisT[i]=Theta004^(i-1)*(-Theta104)^(8-i);
});

\\Equation system
Condition=vector(8);
for(i=1,8,{
  Condition[i]=vector(8);
});
for(i=1,8,{
  \\fS0=0
  Condition[1][i]=polcoef(BasisS[i],0);
  \\fS0.5=0
  Condition[2][i]=polcoef(BasisS[i],1);
  \\fS1=0
  Condition[3][i]=polcoef(BasisS[i],2);
  \\fS1.5=0
  Condition[4][i]=polcoef(BasisS[i],3);
  \\fS2=0
  Condition[5][i]=polcoef(BasisS[i],4);
  \\c=2
  Condition[6][i]=polcoef(Basis[i],0);
});

\\Condition psi(-1/t)+psi(t+1)=psi(t) for the two first coefficients
for(i=7,8,{
  for(j=1,8,
    Condition[i][j]=polcoef(BasisS[j],i)+polcoef(BasisT[j],i)-polcoef(Basis[j],i);
  );
});

M=matconcat([Condition[1];Condition[2];Condition[3];Condition[4];Condition[5];
   Condition[6];Condition[7];Condition[8]]);
A=[0,0,0,0,0,1,0,0]~;

X=matsolve(M,A);

printf("\nThe solution of the system is:");
printf(concat("X=",X));

\\Computation of the series of the modular forms
Delta2Psi=sum(i=1,8,X[i]*Basis[i]);
Delta2PsiS=sum(i=1,8,X[i]*BasisS[i]);
Delta2PsiT=sum(i=1,8,X[i]*BasisT[i]);

Psi=Delta2Psi/Discriminant^2;
PsiS=Delta2PsiS/Discriminant^2;
PsiT=Delta2PsiT/Discriminant^2;

printf("\nThe Fourier series of psi(t) and psi(-1/t) are (r=q^1/2):");
```

```
printf(concat("\npsi(t)=",Psi+O(q^nDisplay)));
printf(concat("\npsi(-1/t)=",PsiS+O(q^nDisplay)));

printf("\nThe error on the equation psi(-1/t)+psi(t+1)=psi(t) is at most:");
printf(concat("psi(-1/t)+psi(t+1)-psi(t)=",PsiS+PsiT-Psi));
```

References

[1] Henry Cohn and Noam Elkies. New upper bounds on sphere packings. I. *Ann. of Math. (2)*, 157(2):689–714, 2003.

[2] Maryna S. Viazovska. The sphere packing problem in dimension 8. *Ann. of Math. (2)*, 185(3):991–1015, 2017.

[3] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska. The sphere packing problem in dimension 24. *Ann. of Math. (2)*, 185(3):1017–1033, 2017.

[4] Thomas C. Hales. A proof of the Kepler conjecture. *Ann. of Math. (2)*, 162(3):1065–1185, 2005.

[5] Thomas Hales, Mark Adams, Gertrud Bauer, Tat Dat Dang, John Harrison, Le Truong Hoang, Cezary Kaliszyk, Victor Magron, Sean McLaughlin, Tat Thang Nguyen, Quang Truong Nguyen, Tobias Nipkow, Steven Obua, Joseph Pleso, Jason Rute, Alexey Solovyev, Thi Hoai An Ta, Nam Trung Tran, Thi Diep Trieu, Josef Urban, Ky Vu, and Roland Zumkeller. A formal proof of the Kepler conjecture. *Forum Math. Pi*, 5:e2, 29, 2017.

[6] L. Fejes. Über die dichteste Kugellagerung. *Math. Z.*, 48:676–684, 1943.

[7] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov.

[8] Henry Cohn. A conceptual breakthrough in sphere packing. *Notices Amer. Math. Soc.*, 64(2):102–115, 2017.

[9] Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna Viazovska. Universal optimality of the E_8 and Leech lattices and interpolation formulas. *arXiv e-prints*, page arXiv:1902.05438, Feb 2019.

[10] Manfred Einsiedler and Thomas Ward. *Functional analysis, spectral theory, and applications*, volume 276 of *Graduate Texts in Mathematics*. Springer, Cham, 2017.

[11] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[12] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[13] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[14] Kathrin Bringmann, Amanda Folsom, Ken Ono, and Larry Rolen. *Harmonic Maass forms and mock modular forms: theory and applications*, volume 64 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2017.

[15] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer, Berlin, sixth edition, 2018. See corrected reprint of the 1998 original [MR1723092], Including illustrations by Karl H. Hofmann.

[16] David Mumford. *Tata lectures on theta. I.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition.

[17] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006.